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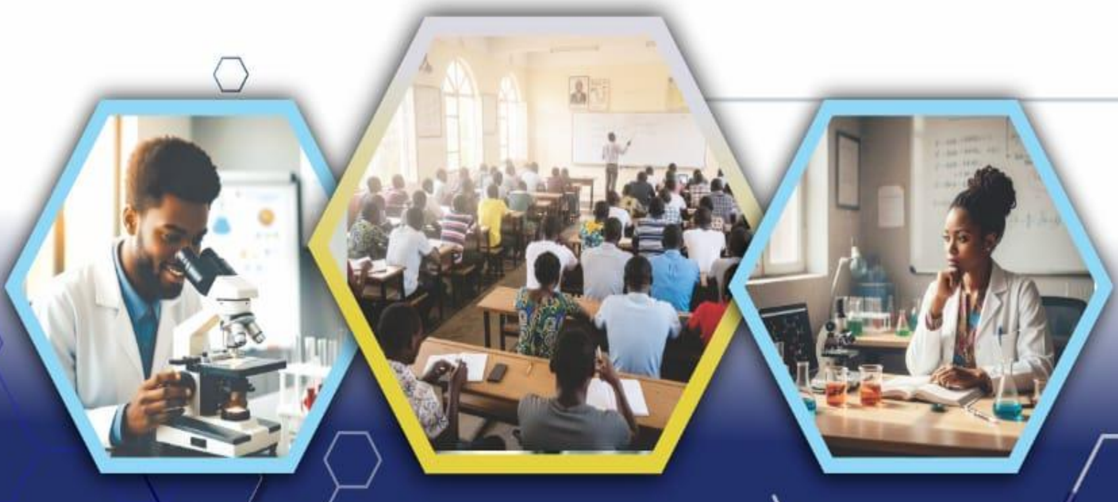

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STRONG CONVERGENCE OF PENTAGONAL ITERATIVE SCHEME FOR A GENERAL CLASS OF FUNCTIONS

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Abstract
In this paper, we employ the notion of a general class of functions, introduced by Bosede (2011), to establish the strong convergence of the pentagonal iterative scheme in a complete normed linear space. As special cases, we also prove the strong convergence of the Noor, Ishikawa, and Mann iterative schemes. Our results generalize, improve, and unify several well-known results in literature.

Keywords: Pentagonal Iterative Scheme, Strong Convergence, Banach Space, Fixed Point

Introduction

Let (B, d) be a complete metrics space, $G: B \to B$ a selfmap of B and $G_T = \{p \in B : Gp = p\}$ the set of fixed points of T in B.

Let $\{x_n\}_{n=0}^{\infty} \subset$ B E be a sequence generated by an iteration procedure involving the operator T, that is,

$$x_{n+1} = f(G, x_n), \text{ n} = 0, 1, 2, \ldots, \qquad (1.1)$$

Where $x_0 \in B$ is the initial approximation and f is some function setting.

$$f(G, x_n) = Gx_n, \text{ n} = 0, 1, 2, \ldots, \qquad (1.2)$$

In (1.1), we have the Picard iteration process,

Putting

$$f(G, x_n) = (1 - \alpha_n)x_n + \alpha_n G x_n, \text{ n} = 0.1.2, \ldots, \qquad (1.3)$$

In (1.1), where $\{x_n\}_{n=0}^{\infty}$ is a sequence of real number in [0,1], we have the Mann iteration process Mann, (1953).

For $x_0 \in B$, the sequences $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n G w_n$$

$$w_n = (1-\beta_n)x_n + \beta_n G v_n \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequence of real numbers in [0,1], is called the Ishikawa iteration process. [For examples, see Ishikawa (1974)].

Also:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n G w_n$$

$$w_n = (1-\beta_n)x_n + \beta_n G v_n$$

$$v_n = (1-\gamma_n)x_n + \gamma_n G y_n \tag{1.5}$$

Where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequence of real numbers in [0,1], is called the Noor iteration scheme (1988).

For arbitrary $x_0 \in B$, let be the Pentagonal Iterative Scheme defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n G w_n$$

$$w_n = (1-\beta_n)x_n + \beta_n G v_n$$

$$v_n = (1-\gamma_n)x_n + \gamma_n G y_n$$

$$y_n = (1-\theta_n)x_n + \theta_n G z_n$$

$$z_n = (1-\delta_n)x_n + \delta_n G x_n \tag{1.6}$$

Where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ $\{\theta_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are sequence of real numbers in [0,1], is called the Pentagonal iterative scheme.

Preliminaries

Many authors including Bosede and Rhoade (2010), Bosede (2009), Bosede (2010) and Bosede (2011) employed the Zamfirescu condition to establish several interesting convergence results for Mann, Ishikawa and Noor iteration processes in a uniformly convex Banach space.

The result of Rhoades (1974 and 1976) were also extended by Berinde (2004) to an arbitrary Banach space for the same fixed point iteration processes. Several other researchers such as Bosede (2010 and 2009) and Rafiq (2006) obtained some interesting convergence results for some iteration procedures using various contractive definitions.

Employing a new idea, Osilike (1994/1996) considered the following contractive definition: there exist L>0, $a \in [0,1)$ such that for each $x, y \in B$

$$d(Gx, Gy) \leq Ld(x, Gx) + ad(x, y) \tag{2.1}$$

and established G-stability for such maps with respect to Picards, Kirk, Mann, Ishikawa and Noor iterations.

Imoru and Olatinwo (2003) later extended the results of osilike (1995/1996) and proved some stability result for Picard and Mann iteration processes using the following contractive condition: there exist $b \in [0,1)$ and a monotone increasing function $\varphi : \mathcal{R}^+ \to \mathcal{R}^+$ with $\varphi(0) = 0$ such that for each $x, y \in B$

$$d(Gx, Gy) \leq \varphi(d(x, Gx)) + bd(x, y)$$

(2.2)

A lot of "generalization' and contraction condition similar to (2.2) were also employed by several authors especially Olatinwo (2008) to establish strong convergence results for some iteration processes. For examples, see Imoru and Olatinwo (2003) and Olatinwo (2008).

In 2010, Bosede and Rhoades (2010) observed that the process of "generalizing" (2.1) could continue ad infinitum as a result of this observation, Bosede and Rhoades (2010) introduced the notion of a general class of function to prove the stability of Picard and Mann iterations. [For examples, see Bosede and Rhoades (2010)]. Our aims in this paper is to prove the pentagonal iterative scheme using the notion of a general class of function considered in Banach spaces. We establish the strong convergences of Ishikawa, Mann iteration and Noor iteration as corollaries.

In the sequel, we shall employ following contractive definition: let (E, ||.||) be a branch space, G: B→ B a Self-map of B, with a fixed point p such that for each y E and 0 ≤ a <1, we have

$$\|p - Gy\| \leq a \|p - y\| \qquad (2.3)$$

Remark 2.1: The contractive condition (2.3) is more general than those considered by Imoru and Olatinwo (2003), Osilike (1995/1996) and several others in the following sense:

By replacing L in (2.1) with more complicated expressions, the process of generalizing (2.1) could continue ad infinitum.

In this paper, We makes an obvious assumption implied by (2.1), and one which render all "generalizations" of the form (2.2) unnecessary.

Furthermore, the condition "α(0) = 0" usually imposed by imoru and Olatinwo (2003) in the contractive definition (2.2) is no longer necessary in our contraction condition (2.3) and this is a further improvement to several known result literature.

Results

Theorem 3.1

Let $(B, \|.\|)$ be a complete normed linear space, $G\colon B \to B$ a selfmap of B with a fixed point p, satisfying the contractive condition (2.3). For $x_0 \in B$, let $\{x_n\}_{n=0}^{\infty}$ be the pentagonal iterative scheme defined by (1.6) converging to p, (that is , $Gp = p$ where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ $\{\theta_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are sequence of real numbers in [0,1], such that $\sum_{k=0}^{\infty} \alpha_k = \infty$. then, the Pentagonal iterative scheme converges strongly p.

Proof:

Using the Pentagonal Iterative Scheme (1.6), the contractive condition (2.3) and the triangle inequality, we have

$$\| x_{n+1} - P \| = \|((1-\alpha_n)x_n + \alpha_n Gw_n - P\|$$

$$=\|((1-\alpha_n)x_n + \alpha_n Gw_n - ((1-\alpha_n) + \alpha_n)P\|$$

$$=\|((1-\alpha_n)x_n + \alpha_n Gw_n - (1-\alpha_n)P - \alpha_n P\|$$

$$=\|((1-\alpha_n)(x_n - P) + \alpha_n (Gw_n - P)\|$$

$$\leq (1-\alpha_n)\|x_n - P\| + \alpha_n\| Gw_n - P \|$$

$$\leq (1-\alpha_n)\|x_n - P\| + a\alpha_n\| w_n - P \| \qquad 3.1$$

Considering the estimate of $\| w_n - P \|$ in 1.1

$$\| w_n - P \| = \| (1- \beta_n)\, x_n + \beta_n\, G v_n - P \|$$

$$= \|(1-\beta_n)x_n + \beta_n G v_n - ((1-\beta_n) + \beta_n)P\|$$

$$= \|(1-\beta_n)x_n + \beta_n G v_n - (1-\beta_n)\, P - \beta_n\, P \|$$

$$= \|(1-\beta_n)(x_n - P) + \beta_n(G v_n - P)\|$$

$$\le (1-\beta_n)\|x_n - P\| + \beta_n \| G v_n - P)\|$$

$$\le (1-\beta_n)\|x_n - P\| + a\beta_n \| v_n - \mathrm{P})\| \tag{3.2}$$

Substitute 1.2 into 1.1 yields

$$\| x_{n+1} - P \| = (1-\alpha_n)\|x_n - P\| + \alpha_n a \| w_n - P \|$$

$$= (1-\alpha_n)\|x_n - P\| + \alpha_n a[(1-\beta_n)\|x_n - P\| + \beta_n\, a \| v_n - \mathrm{P})\|]$$

$$= (1-\alpha_n)\|x_n - P\| + \alpha_n a(1-\beta_n)\|x_n - P\| + \alpha_n\beta_n\, a^2 \| v_n - \mathrm{P})\|$$

$$\le [1 - \alpha_n + \alpha_n a(1-\beta_n) + \alpha_n\beta_n\, a^2] \| v_n - \mathrm{P} \|$$

$$= [1 - \alpha_n + \alpha_n a - \alpha_n\beta_n a + \alpha_n\beta_n\, a^2]\, \| v_n - \mathrm{P} \|$$

$$= [1 - (1-a)\alpha_n - (1-a)\alpha_n\beta_n a]\, \| v_n - \mathrm{P} \|$$

$$= [1 - (1-a)(\alpha_n + \alpha_n\beta_n a)]\, \| v_n - \mathrm{P} \| \tag{3.3}$$

Considering the estimate of $\| V_n - P \|$ in 1.3

$$\|V_n - P\| = \|(1-\gamma_n)\, x_n + \gamma_n\, G y_n - P\|$$

$$= \|(1-\gamma_n)x_n + \gamma_n\, G y_n - ((1-\gamma_n) + \gamma_n)\, P)\|$$

$$= \|(1-\gamma_n)x_n + \gamma_n\, G y_n - (1-\gamma_n)P - \gamma_n P\|$$

$$= \| (1-\gamma_n)\, (x_n - P) + \gamma_n\, (G y_n - P)\|$$

$$\le (1-\gamma_n)\, \| x_n - P\| + \gamma_n\, \| G y_n - P\|$$

$$\le (1-\gamma_n)\, \| x_n - P\| + a\gamma_n\, \| y_n - P\| \tag{3.4}$$

Substitute 1.4 into 1.3 yields

$$\| x_{n+1} - P \| = [1 - (1-a)(\alpha_n + \alpha_n\beta_n \mathrm{a})]\, \| v_n - \mathrm{P} \|$$

$$= [1 - (1-a)\alpha_n - (1-a)\alpha_n\beta_n a][\, (1-\gamma_n)\, \| x_n - P\| + \gamma_n\, a \| y_n - P)\|]$$

$$\le [1 - (1-a)\alpha_n - (1-a)\alpha_n\beta_n a]\, (1-\gamma_n + \gamma_n\, a) \| y_n - P)\|] \tag{3.5}$$

Considering the estimate of $\| y_n - P \|$ in 1.5

$$\| y_n - P \| = \|(1-\theta_n)x_n + \theta_n G z_n - P \|$$

$$= \|(1-\theta_n)x_n + \theta_n G z_n - ((1-\theta_n) + \theta_n)\, P)\|$$

$$= \|(1-\theta_n)x_n + \theta_n G z_n - (1-\theta_n)P - \theta_n P\|$$

$$= \| (1 - \theta_n) (x_n - P) + \theta_n (Gz_n - P)\|$$

$$\leq (1 - \theta_n) \| x_n - P\| + \theta_n \| Gz_n - P\|$$

$$\leq (1 - \theta_n) \| x_n - P\| + a\theta_n \| z_n - P\| \qquad 3.6$$

Substitute 1.6 into 1.5 yields

$$\| x_{n+1} - P \| = [1 - (1 - a)(\alpha_n + \alpha_n\beta_n a)][(1 - \gamma_n + \gamma_n a)\|y_n - P\|]$$

$$= [(1 - (1 - a)\alpha_n - (1 - a)\alpha_n\beta_n a)][(1 - \gamma_n + \gamma_n a)((1 - \theta_n)\|x_n - P\| + a\theta_n \|z_n - P\|)]$$

$$\leq [(1 - (1 - a)\alpha_n - (1 - a)\alpha_n\beta_n a)][(1 - \gamma_n + \gamma_n a)(1 - \theta_n + a\theta_n)\|z_n - P\|] \qquad 3.7$$

Similarly, $\| z_n - P\|$ in 1.7 is estimated as follows:

$$\| z_n - P\| = \|(1 - \delta_n)x_n + \delta_n Gx_n - \mathrm{P}\|$$

$$= \|(1 - \delta_n)x_n + \delta_n Gx_n - ((1 - \delta_n) + \delta_n)\mathrm{P}\|$$

$$= \|(1 - \delta_n)x_n + \delta_n Gx_n - (1 - \delta_n)\mathrm{P} - \delta_n\mathrm{P}\|$$

$$= \|(1 - \delta_n)(x_n - P) + \delta_n(Gx_n - \mathrm{P})\|$$

$$\leq (1 - \delta_n)\|x_n - P \| + \delta_n\|Gx_n - \mathrm{P}\|$$

$$\leq (1 - \delta_n)\|x_n - P \| + \delta_n \mathrm{a}\|x_n - \mathrm{P}\|$$

$$= (1 - \delta_n + \delta_n\mathrm{a}) \|x_n - \mathrm{P}\| \qquad 3.8$$

Substitute 1.8 into 1.7

$$\| x_{n+1} - P \| \leq [(1 - (1 - a)\alpha_n - (1 - a)\alpha_n\beta_n a)][(1 - \gamma_n + \gamma_n a)(1 - \theta_n + a\theta_n)\| z_n - P\|]$$

$$= [(1 - (1 - a)\alpha_n - (1 - a)\alpha_n\beta_n a)][(1 - \gamma_n + \gamma_n a)(1 - \theta_n + a\theta_n)(1 - \delta_n + \delta_n\mathrm{a}) \|x_n - \mathrm{P}\|]$$

$$\leq 1 - (1 - a)\alpha_n \|x_n - \mathrm{P}\|$$

$$\leq \prod_{k=0}^{n}[1 - (1 - a)\alpha_k] \|x_0 - \mathrm{P}\| \qquad 3.9$$

$$\leq \prod_{k=0}^{n} e^{-(1-a)\alpha_k} \|x_0 - \mathrm{P}\|$$

$$= e^{-(1-a)\alpha_k} \sum_{k=0}^{n} \alpha_k \|x_0 - P\| \to 0$$

as n → ∞. Since $\sum_{k=0}^{n} \alpha_k = \infty$, $a \in [0,1)$ and from (3.9), we have $\|x_n - \mathrm{P}\| \to 0$ as n → ∞, which implies that the pentagonal iterative scheme process converges strongly to p.

To prove the uniqueness, we take $p, q \in F_G$,where F_G is the set of fixed points of G in B such that $p = Gp$ and $q = Gq$.
Suppose on the contrary that $p \neq q$. Then, using the contractive condition (2.3) and since $0 \leq a < 1$, we have

$$\begin{aligned} \|p - q\| &= \|p - Gq\| \\ &\leq a\|p - q\| \qquad 3.10 \\ &< \|p - q\| \end{aligned}$$

which is a contradiction. Therefore, $p = q$.

This completes the proof.

Consequently, we have the following corollaries:

Corollary 3.2.
Let $(B, \|.\|)$ be a complete normed linear space, $G: B \to B$ a selfmap of B with a fixed point p, satisfying the contractive condition (2.3). For $x_0 \in B$, let $\{x_n\}_{n=0}^{\infty}$ be Mann iterative scheme defined by (1.3) converging to p, (that is , $Gp = p$) where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in [0,1], such that $\sum_{k=0}^{\infty} \alpha_k = \infty$. Then, Mann iterative scheme converges strongly p.

Corollary 3.3.
Let $(B, \|.\|)$ be a complete normed linear space, $G: B \to B$ a selfmap of B with a fixed point p, satisfying the contractive condition (2.3). For $x_0 \in B$, let $\{x_n\}_{n=0}^{\infty}$ be Ishikawa iteration defined by (1.4) converging to p, (that is , $Gp = p$) where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1], such that $\sum_{k=0}^{\infty} \alpha_k = \infty$. Then, Ishikawa iteration process converges strongly p.

Corollary 3.4.
Let $(B, \|.\|)$ be a complete normed linear space, $G: B \to B$ a selfmap of B with a fixed point p, satisfying the contractive condition (2.3). For $x_0 \in B$, let $\{x_n\}_{n=0}^{\infty}$ be Noor iterative scheme defined by (1.5) converging to p, (that is , $Gp = p$) where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1], such that $\sum_{k=0}^{\infty} \alpha_k = \infty$. then, Noor iterative scheme converges strongly p.

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