



JOURNAL OF SCIENCE, TECHNOLOGY AND EDUCATION (JSTE)



**A PUBLICATION OF THE
DEPARTMENT OF SCIENCE,
TECHNOLOGY AND MATHEMATICS
EDUCATION (STME),
NASARAWA STATE UNIVERSITY, KEFFI, NIGERIA**



VOLUME 10

ISSN: 2651-5539

ANALYTICAL APPROXIMATION OF LYAPUNOV ORBITS NEAR COLLINEAR EQUILIBRIA IN A RADIATING AND OBLATE α -CENTAURI MODEL

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Citation: Audu, A. O., Adigun, M. K., & Popoola, S. O. (2026). Homoclinic trajectories and periodic-orbit families near l_3 in the planar photogravitational restricted three-body problem. *Journal of Science, Technology, and Education (JSTE)*; www.nsukjste.com/. 10(43), 590-613.

Abstract

Collinear equilibrium points organise important families of local trajectories in restricted three-body dynamics, but the inclusion of radiation pressure and oblateness complicates both their location and the construction of periodic approximations. This study examines the planar circular restricted three-body formulation adopted for the α -Centauri binary system, with dimensionless mass parameter $\mu = 0.47333$, following the parameter convention of Singh and Gyegwe (2017), and radiating, oblate primaries. The equations of motion and collinear equilibrium conditions are stated in a rotating frame; local Taylor expansions about the three equilibria provide the basis for a Lindstedt–Poincaré construction that incorporates an amplitude-dependent time transformation. Particular attention is paid to the cubic-order Coriolis terms and the relation between time scaling and velocity reconstruction. Six reported combined parameter configurations are compared using equilibrium coordinates and

benchmark starting conditions, periods, Jacobi constants and frequencies. For example, the reported L_1 equilibrium coordinate changes from -1.20751483 to -0.79423688 between the first two cases, while the corresponding values at the remaining equilibria also change non-monotonically. Because radiation and oblateness vary simultaneously, these differences cannot be attributed uniquely to either effect. Moreover, the numerical tables are benchmarks inherited from earlier work, not new integrations of the present approximation. The formulation provides a structured basis for examining perturbative consistency and designing subsequent numerical verification; confirmation of orbit closure, invariant preservation and predictive accuracy remains necessary.

Keywords: planar restricted three-body problem; α -Centauri; collinear libration points; Lyapunov orbits; radiation pressure; oblateness; Lindstedt–Poincaré approximation.

1. Introduction

The circular restricted three-body problem (CR3BP) idealises the motion of a massless third body in the rotating gravitational field of two primaries. The simplification retains a rich equilibrium and periodic-orbit structure, providing a tractable starting point for investigating nonlinear motion (Jorba-Cuscó et al., 2018; Owen & Baresi, 2024). Its five equilibria include three collinear and two triangular points. The collinear regions are especially important because nearby centre and hyperbolic dynamics permit local oscillatory motions while making their numerical continuation and dynamical interpretation delicate (Celletti et al., 2024; Lin et al., 2024).

Periodic families emerging near collinear equilibria have been studied using local expansions, differential correction and dynamical-systems methods. Richardson (1980) developed a third-order analytical representation of three-dimensional halo-type motion, whereas Howell (1984) investigated numerical families and their stability properties. Subsequent work on centre manifolds and invariant structures demonstrated that an analytically plausible periodic approximation and a numerically closed orbit are distinct objects requiring separate tests (Doan, 1985; Jorba-Cuscó et al., 2018). The Lyapunov families are therefore best understood in conjunction with

the surrounding nonlinear phase-space geometry rather than only through selected initial conditions.

The role of collinear motion extends beyond constructing individual periodic trajectories. Invariant manifolds associated with Lyapunov families organise transfers and transport near the primaries, as established in the geometric investigation of transverse homoclinic intersections (Barrabés et al., 2017; Delshams et al., 2025). Large-neighbourhood computations later described coexisting periodic solutions and invariant tori, providing a more detailed view of the motion around all three collinear points (Burgos-García & Gidea, 2015). This literature makes local expansions scientifically useful, but also sets a clear standard for verification: phase-space behaviour cannot be inferred solely from equilibrium-coordinate plots.

The classical CR3BP omits effects that can matter for an idealised photogravitational model. Radiation pressure modifies the effective attraction associated with a luminous primary, while an oblateness correction accounts for departures from spherical symmetry. Earlier mathematical studies of nonspherical primaries and photogravitational equilibria found that these perturbations may shift equilibrium locations and alter the form of characteristic equations (Kumar & Kumar, 2026; Saputra et al., 2026;

Zotos, 2015). The physical scope of each modification depends on the adopted scaling, the assumptions underlying the radiation factors, and the order in which nonspherical terms are retained.

Investigations of combined radiation, oblateness and additional force perturbations further show why the effects must not be conflated. Kushvah analysed equilibrium configurations in a photogravitational model that also addressed Poynting–Robertson drag (Kushvah, 2008a, 2008b). AbdulRaheem and Singh (2008), examined periodic motion around triangular equilibrium points in a perturbed restricted three-body formulation. These studies clarify that observations in which multiple model coefficients change together do not constitute controlled evidence of the separate contribution of any one coefficient. The distinction is crucial for the six configurations tabulated in the present α -Centauri application.

More recent extensions examine elliptic primary motion, triaxial bodies and belt potentials, which introduce modelling assumptions not present in the circular, oblate formulation used here. Singh and Isah (2021), examined collinear configurations in an elliptic restricted three-body model incorporating radiating and triaxial primaries together with an additional belt potential. Singh and Ashagwu explored drag, oblateness and radiation in an elliptic

formulation, albeit with primary emphasis on triangular points (Arif et al., 2023; Singh & Ashagwu, 2024). These investigations are relevant comparisons, but their additional forcing terms mean that their numerical results cannot be applied directly to the present circular equations.

Analytical perturbation theory provides another line of development. The Lindstedt method controls secular terms by modifying the time variable or frequency order by order; the method has particular advantages when periodic solutions are sought in weakly nonlinear oscillatory systems (Craps et al., 2014; Krishnaswami & Senapati, 2019). For restricted three-body dynamics, such an expansion must be linked carefully to the Coriolis coupling, local potential derivatives and the transformation back to physical time. Otherwise, formally small inconsistencies can propagate into the order at which the period and initial velocity are reported.

Higher-order and mixed analytic–numerical studies indicate that order alone is not evidence of improved predictive performance. Sheth et al. (2021) reported fifth-order halo-orbit approximations for radiating and oblate primaries, along with a differential-correction comparison (Amato et al., 2019; Philipp et al., 2018; Salazar et al., 2020). Paez and Guzzo (2022), explored semi-analytical halo structures in an elliptic setting (Amato et al., 2019; Bini et al., 2019),

while studies of radiatively forced artificial halo motion address effects not captured by a purely gravitationally corrected planar model (Hristov, 2026; Lara et al., 2024). Across these approaches, a credible accuracy claim typically requires residual or closure measures rather than agreement between plots based on visual inspection.

The present study builds on an earlier α -Centauri numerical investigation by Singh and Gyegwe (2017), which reported six parameter configurations comprising equilibrium coordinates, initial positions, transverse velocities, orbital periods, Jacobi constants and frequencies. Building on this foundation, the present study develops a mathematical formulation and analysis of the collinear-equilibrium conditions, local Taylor expansions and Lindstedt–Poincaré perturbation equations. MATLAB software is employed to generate graphical representations and examine variations in equilibrium locations and Lyapunov-orbit characteristics across the radiation-pressure and oblateness parameter configurations considered. This approach integrates mathematical modelling with numerical and graphical analysis to investigate the behaviour of near-equilibrium orbital configurations in the perturbed α -Centauri system. The mathematical analysis addresses two principal considerations. First, the

collinear-equilibrium and Taylor-expansion equations are formulated consistently with the coordinate convention placing the primaries at $(\mu, 0)$ and $(\mu - 1, 0)$. Second, the amplitude-dependent time transformation, $t = \kappa\tau$ with $\kappa = 1 + \rho\epsilon^2$, is incorporated into the Lindstedt–Poincaré formulation, accounting for its effects on the Coriolis terms and reconstructed velocities. The analysis examines the equilibrium locations $L_1 - L_3$, develops the local perturbative relations and evaluates the variations in orbital parameters across six combined radiation-pressure and oblateness configurations. The resulting mathematical and graphical representations provide a basis for interpreting the associated Lyapunov-orbit characteristics, while independent verification of orbital closure, stability and numerical convergence remains necessary.

2. Methodology

2.1 Model formulation

The planar circular restricted three-body model has primaries of dimensionless masses $1 - \mu$ and μ located at $(\mu, 0)$ and $(\mu - 1, 0)$, respectively, following the coordinate convention adopted by Singh and Gyegwe (2017). Both primaries are oblate and radiating. The massless particle moves in the rotating (x, y) frame. The equations of motion are:

$$\ddot{x} - 2n\dot{y} = \Omega_x, \quad \ddot{y} + 2n\dot{x} = \Omega_y \quad (1)$$

$$\Omega = \frac{1}{2}n^2(x^2 + y^2) + (1 - \mu)q_1/r_1 + \mu q_2/r_2 + (1 - \mu)A_1q_1/(2r_1^3) + \mu A_2q_2/(2r_2^3) \quad (2)$$

In the adopted formulation, the radiation reduction factors q_i are incorporated into both the inverse-distance gravitational terms and the corresponding oblateness corrections, consistent with the photogravitational model discussed by Abouelmagd (2013). This formulation accounts for the combined contributions of radiation pressure and oblateness to the effective potential and is applied consistently in deriving the collinear equilibrium equations, local Taylor expansions and subsequent Lindstedt–Poincaré perturbation equations.

$$r_1 = \sqrt{[(x - \mu)^2 + y^2]}, \quad r_2 = \sqrt{[(x + 1 - \mu)^2 + y^2]} \quad (3)$$

$$\mu = m_2/(m_1 + m_2), \quad n^2 = 1 + 3(A_1 + A_2)/2 \quad (4)$$

$$C = 2\Omega - (\dot{x}^2 + \dot{y}^2) \quad (5)$$

Here q_i are radiation reduction factors, A_i are oblateness coefficients, n is the mean motion, and C is the Jacobi constant.

2.2 Collinear equilibrium equation

Collinear equilibrium coordinates satisfy $y = 0$ and $\Omega_x = 0$. Differentiation of the stated potential yields

$$F(x) = n^2x - (1 - \mu)q_1(x - \mu)/r_1^3 - \mu q_2(x + 1 - \mu)/r_2^3 - 3(1 - \mu)A_1q_1(x - \mu)/(2r_1^5) - 3\mu A_2q_2(x + 1 - \mu)/(2r_2^5) = 0 \quad (6)$$

$$F'(x) = n^2 + (1 - \mu)q_1[3(x - \mu)^2/r_1^5 - 1/r_1^3] + \mu q_2[3(x + 1 - \mu)^2/r_2^5 - 1/r_2^3] + 3(1 - \mu)A_1q_1[5(x - \mu)^2/r_1^7 - 1/r_1^5]/2 + 3\mu A_2q_2[5(x + 1 - \mu)^2/r_2^7 - 1/r_2^5]/2 \quad (7)$$

Each root is sought in one of the disjoint intervals $(-\infty, \mu - 1)$, $(\mu - 1, \mu)$, and (μ, ∞) . A bracketed root-finding method avoids crossing singularities at the primaries. The labels L_1 , L_2 and L_3 are retained as reported in the source tables, although this paper's coordinate ordering differs from some conventional nomenclature.

2.3 Linearization and local Taylor coefficients

For a selected collinear equilibrium xL , introduce local displacements $\varphi = x - xL$ and $\psi = y$. Subscripts on Ω denote partial derivatives evaluated at $(xL, 0)$. Symmetry under $y \rightarrow -y$ eliminates the odd powers of ψ in Ω_x and the even powers in Ω_y .

$$\ddot{\varphi} - 2n\dot{\psi} = h_1\varphi + h_2\varphi^2 + h_3\varphi^3 + h_4\psi^2 + h_5\varphi\psi^2 + O(4) \quad (8)$$

$$\ddot{\psi} + 2n\dot{\varphi} = D_1\psi + D_2\varphi\psi + D_3\varphi^2\psi + D_4\psi^3 + O(4) \quad (9)$$

$$\begin{aligned} h_1 &= \Omega_{xx}, & h_2 &= \frac{1}{2}\Omega_{xxx}, & h_3 &= (1/6)\Omega_{xxxx}, \\ h_4 &= \frac{1}{2}\Omega_{xyy}, & h_5 &= \frac{1}{2}\Omega_{xxyy} \end{aligned} \quad (10)$$

$$D_1 = \Omega_{yy}, \quad D_2 = \Omega_{xyy}, \quad D_3 = \frac{1}{2}\Omega_{xxyy}, \quad D_4 = (1/6)\Omega_{yyy} \quad (11)$$

$$\begin{aligned} h_1 &= n^2 + 2(1 - \mu)q_1/r_1^3 + 2\mu q_2/r_2^3 + 6(1 - \mu)A_1q_1/r_1^5 \\ &\quad + 6\mu A_2q_2/r_2^5 \end{aligned} \quad (12)$$

$$\begin{aligned} D_1 &= n^2 - (1 - \mu)q_1/r_1^3 - \mu q_2/r_2^3 \\ &\quad - 3(1 - \mu)A_1q_1/(2r_1^5) - 3\mu A_2q_2/(2r_2^5) \end{aligned} \quad (13)$$

The coefficient definitions are stated once, rather than repeated through separate derivative subsections. Higher-order coefficients are obtained by differentiating the same potential and must be evaluated at each equilibrium.

2.4 Lindstedt–Poincaré approximation

A small amplitude ε is introduced with time rescaling $t = \kappa\tau$. The correction $\kappa = 1 + \rho\varepsilon^2$ accounts for amplitude-dependent frequency. Primes below denote derivatives with respect to τ .

$$\varphi = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + O(\varepsilon^4), \quad \psi = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + O(\varepsilon^4) \quad (14)$$

$$\kappa = 1 + \rho\varepsilon^2, \quad \varphi'' - 2n\kappa\psi' = \kappa^2\Omega\varphi, \quad \psi'' + 2n\kappa\varphi' = \kappa^2\Omega\psi \quad (15)$$

At first order, substituting $\varphi_1 = \cos(\omega\tau)$ and $\psi_1 = \beta \sin(\omega\tau)$ gives

$$(\omega^2 + h_1) + 2n\omega\beta = 0, \quad 2n\omega + (\omega^2 + D_1)\beta = 0 \quad (16)$$

$$\beta = -(\omega^2 + h_1)/(2n\omega), \quad (\omega^2 + h_1)(\omega^2 + D_1) - 4n^2\omega^2 = 0 \quad (17)$$

At second order, the right-hand sides are $h_2\varphi_1^2 + h_4\psi_1^2$ and $D_2\varphi_1\psi_1$. Their harmonic components are obtained using $\cos^2\theta = (1 + \cos 2\theta)/2$, $\sin^2\theta = (1 - \cos 2\theta)/2$ and $\sin\theta\cos\theta = \sin 2\theta/2$.

$$F_0 = \frac{1}{2}(h_2 + h_4\beta^2), \quad F_2 = \frac{1}{2}(h_2 - h_4\beta^2), \quad G_2 = \frac{1}{2}D_2\beta \quad (18)$$

$$\varphi_2 = a_0 + a_2\cos(2\omega\tau), \quad \psi_2 = b_2\sin(2\omega\tau), \quad a_0 = -F_0/h_1 \quad (19)$$

$$(4\omega^2 + h_1)a_2 + 4n\omega b_2 = -F_2 \quad (20)$$

$$4n\omega a_2 + (4\omega^2 + D_1)b_2 = -G_2 \quad (21)$$

The cubic-order forcing includes the time-scaling corrections. The resulting governing equations are

$$\begin{aligned} \varphi_3'' - 2n\psi_3' - h_1\varphi_3 &= 2n\rho\psi_1' + 2\rho h_1\varphi_1 + 2h_2\varphi_1\varphi_2 \\ &+ h_3\varphi_1^3 + 2h_4\psi_1\psi_2 + h_5\varphi_1\psi_1^2 \end{aligned} \quad (22)$$

$$\begin{aligned} \psi_3'' + 2n\varphi_3' - D_1\psi_3 &= -2n\rho\varphi_1' + 2\rho D_1\psi_1 \\ &+ D_2(\varphi_1\psi_2 + \varphi_2\psi_1) + D_3\varphi_1^2\psi_1 + D_4\psi_1^3 \end{aligned} \quad (23)$$

To determine the third-order correction, the cubic-order forcing terms are decomposed into fundamental and third harmonics. The elimination of resonant terms provides the solvability condition for the time-scaling parameter ρ . The remaining third-harmonic components determine the coefficients of the periodic approximation.

$$\begin{aligned} P_1 &= 2h_2a_0 + h_2a_2 + \frac{3}{4}h_3 + h_4\beta b_2 + \frac{1}{4}h_5\beta^2 \\ Q_1 &= D_2[\beta a_0 + (b_2 - \beta a_2)/2] + \frac{1}{4}D_3\beta + \frac{3}{4}D_4\beta^3 \end{aligned} \quad (24)$$

$$\rho = -(P_1 + \beta Q_1)/(2h_1 + 4n\omega\beta + 2D_1\beta^2) \quad (25)$$

$$\begin{aligned} P_3 &= h_2a_2 + \frac{1}{4}h_3 - h_4\beta b_2 - \frac{1}{4}h_5\beta^2 \\ Q_3 &= \frac{1}{2}D_2(b_2 + \beta a_2) + \frac{1}{4}D_3\beta - \frac{1}{4}D_4\beta^3 \end{aligned} \quad (26)$$

$$\Delta_3 = (9\omega^2 + h_1)(9\omega^2 + D_1) - 36n^2\omega^2 \quad (27)$$

$$\begin{aligned} c_3 &= [6n\omega Q_3 - (9\omega^2 + D_1)P_3]/\Delta_3 \\ d_3 &= [6n\omega P_3 - (9\omega^2 + h_1)Q_3]/\Delta_3 \end{aligned} \quad (28)$$

The resonant coefficient c_1 is fixed by the amplitude-normalisation convention. Choosing $c_1 = 0$ removes the arbitrary third-order correction parallel to the leading fundamental mode. The

expressions for ρ , c_3 and d_3 require nonzero denominators and are to be evaluated for each parameter case.

Therefore, the third-order approximation is

$$x(\tau) = xL + \varepsilon \cos(\omega\tau) + \varepsilon^2[a_0 + a_2 \cos(2\omega\tau)] + \varepsilon^3[c_1 \cos(\omega\tau) + c_3 \cos(3\omega\tau)] \quad (29)$$

$$y(\tau) = \varepsilon \beta \sin(\omega\tau) + \varepsilon^2 b_2 \sin(2\omega\tau) + \varepsilon^3 d_3 \sin(3\omega\tau) \quad (30)$$

$$\dot{x} = (1/\kappa)dx/d\tau, \quad \dot{y} = (1/\kappa)dy/d\tau, \quad T = 2\pi\kappa/\omega \quad (31)$$

With $c_1 = 0$, the amplitude ε corresponding to a given initial coordinate x_0 satisfies

$$x_0 - xL = \varepsilon + \varepsilon^2(a_0 + a_2) + \varepsilon^3 c_3 \quad (32)$$

The small-amplitude root is selected for each parameter configuration. Case-specific numerical values of ε and independently integrated orbital solutions must be evaluated before quantitative validation against the published benchmark tables.

3. Results

3.1 Equilibrium locations

Table 1. Collinear equilibrium coordinates for α -Centauri ($\mu = 0.47333$).

Case	A ₁	A ₂	q ₁	q ₂	L ₁	L ₂	L ₃
1	0	0	1	1	-1.20751483	-0.03765997	1.18902157
2	0.01	0.001	0.4	0.1	-0.79423688	-0.16548350	0.95833957
3	0.02	0.002	0.5	0.2	-0.88238502	-0.13509348	1.01122572
4	0.03	0.003	0.6	0.3	-0.94520716	-0.12047798	1.05651685
5	0.04	0.004	0.7	0.4	-0.99536901	-0.11247880	1.09634009
6	0.05	0.005	0.8	0.5	-1.03761166	-0.10790176	1.13198424

Note. Illustrative joint radiation–oblateness configurations; Case 1 is unperturbed. Numerical values, Source from Singh and Gyegwe (2017) for comparison.

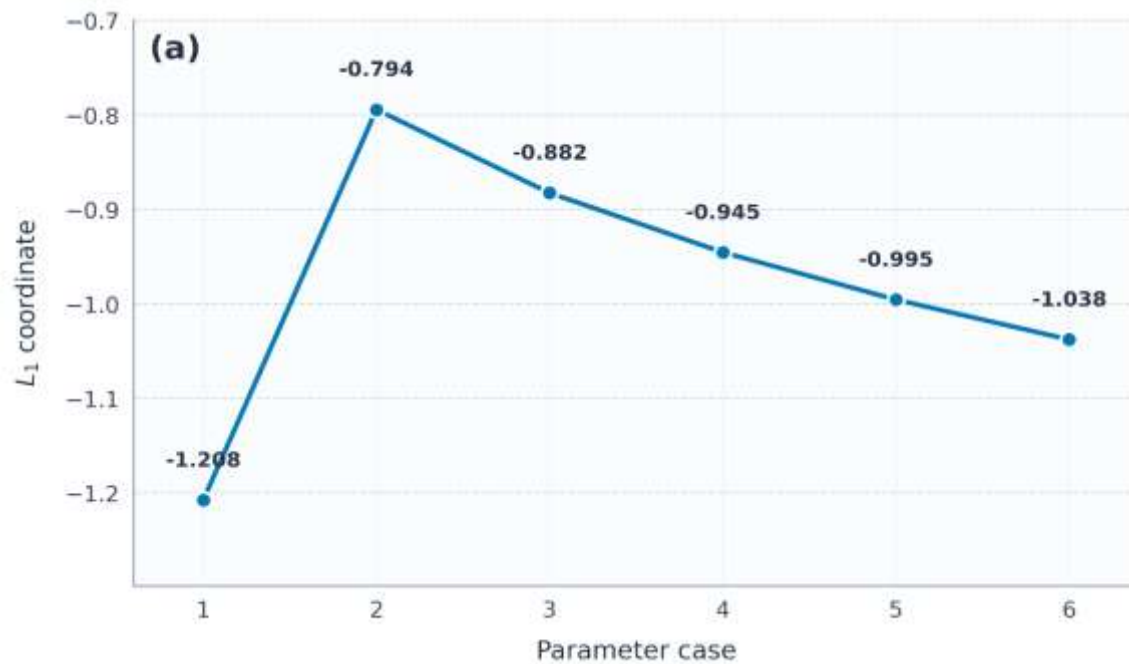


Figure 1(a). Collinear equilibrium coordinate L_1 across parameter Cases 1–6 ($\mu = 0.47333$). The tabulated L_1 coordinate changes from -1.20751483 in Case 1 to -0.79423688 in Case 2, followed by a gradual return toward more negative values through Case 6 (-1.03761166). Case 1 represents the unperturbed reference configuration, whereas Cases 2–6 vary radiation and oblateness parameters together. Consequently, the plotted differences cannot isolate the influence of either perturbation.

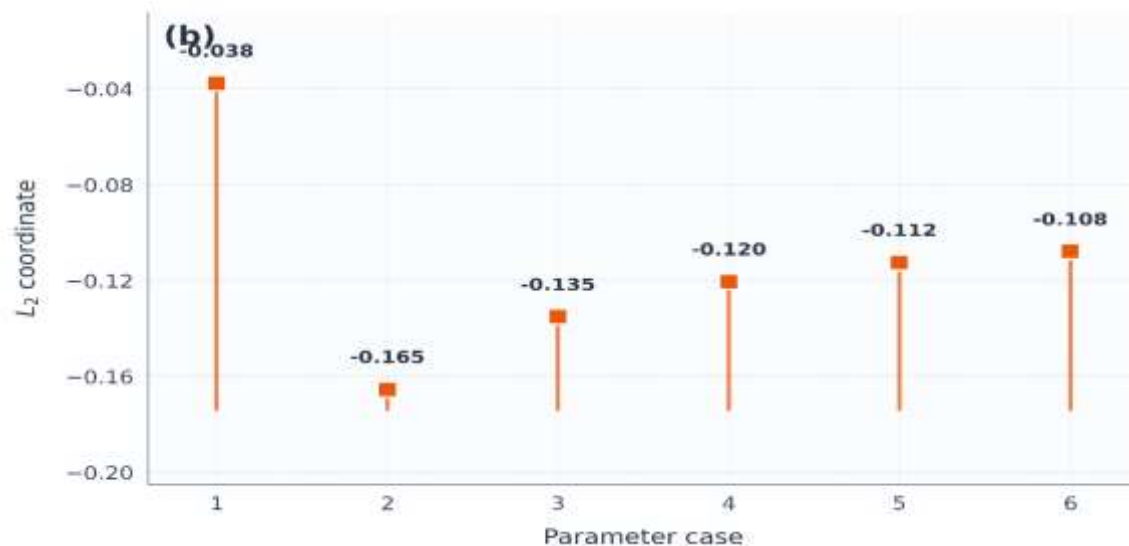


Figure 1(b). Collinear equilibrium coordinate L_2 across parameter Cases 1–6 ($\mu = 0.47333$). The L_2 coordinate becomes more negative between Cases 1 and 2 (-0.03765997 to -0.16548350) and then moves progressively toward -0.10790176 by Case 6. These

movements describe differences among the six combined parameter configurations, without establishing an isolated radiation or oblateness effect.

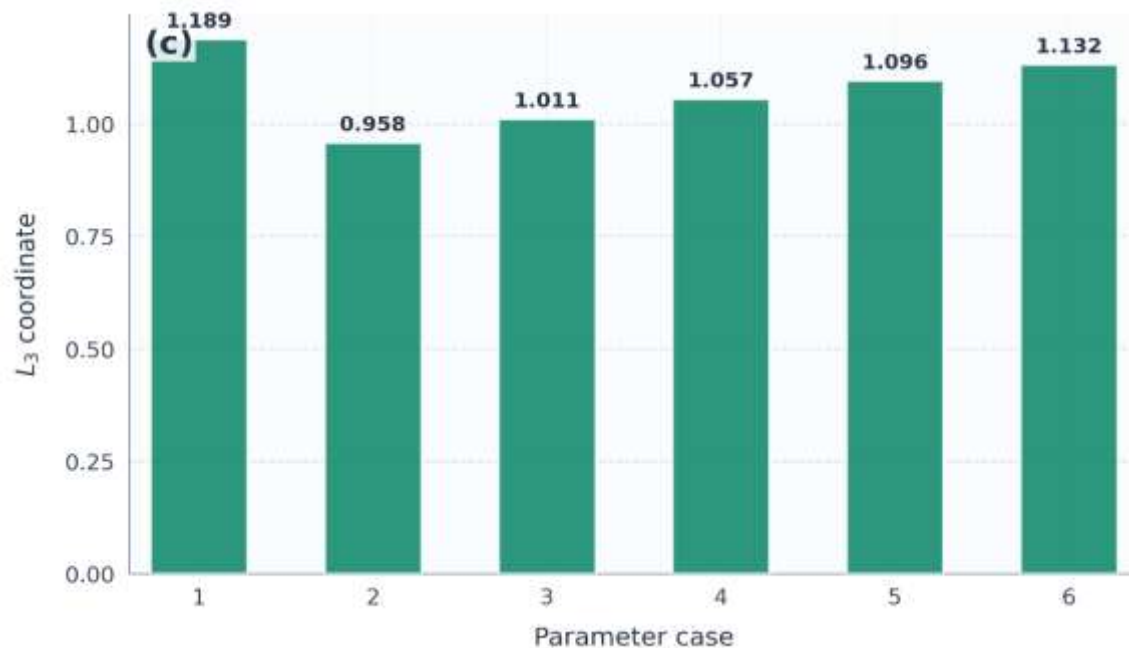


Figure 1(c). Collinear equilibrium coordinate L_3 across parameter Cases 1–6 ($\mu = 0.47333$).

The L_3 coordinate decreases from 1.18902157 in Case 1 to 0.95833957 in Case 2 and then rises through Cases 3–6 to 1.13198424. These results indicate sensitivity of the reported equilibrium location to the combined parameter case.

3.2 Lyapunov-orbit starting conditions

Tables 2–4 present the reported initial x-coordinates, transverse velocities, orbital periods, Jacobi constants and frequencies. These quantities are retained as comparison data for assessing the analytical formulation developed in Section 2; they are not independently integrated periodic-orbit solutions. The MATLAB plots display their variation across the six configurations.

Table 2. Starting conditions for Lyapunov orbits near L_1 ($\mu = 0.47333$).

Case	x_0	$\dot{y}_0$	T	C	ω
1	-1.17772296	-0.15844398	5.26996513	3.47518463	1.19274305
2	-0.76451797	-0.13404904	3.32511205	1.33092657	1.89037668
3	-0.85296994	-0.14128336	3.76434523	1.73135624	1.66980282
4	-0.91603690	-0.15064417	4.03701109	2.07085351	1.55702180
5	-0.96642262	-0.16040252	4.23423349	2.37809889	1.48449874
6	-1.00888380	-0.17018986	4.38698369	2.66454003	1.43281004

Note. Reported starting conditions and orbital descriptors Numerical values, Source from Singh and Gyegwe (2017) for comparison.

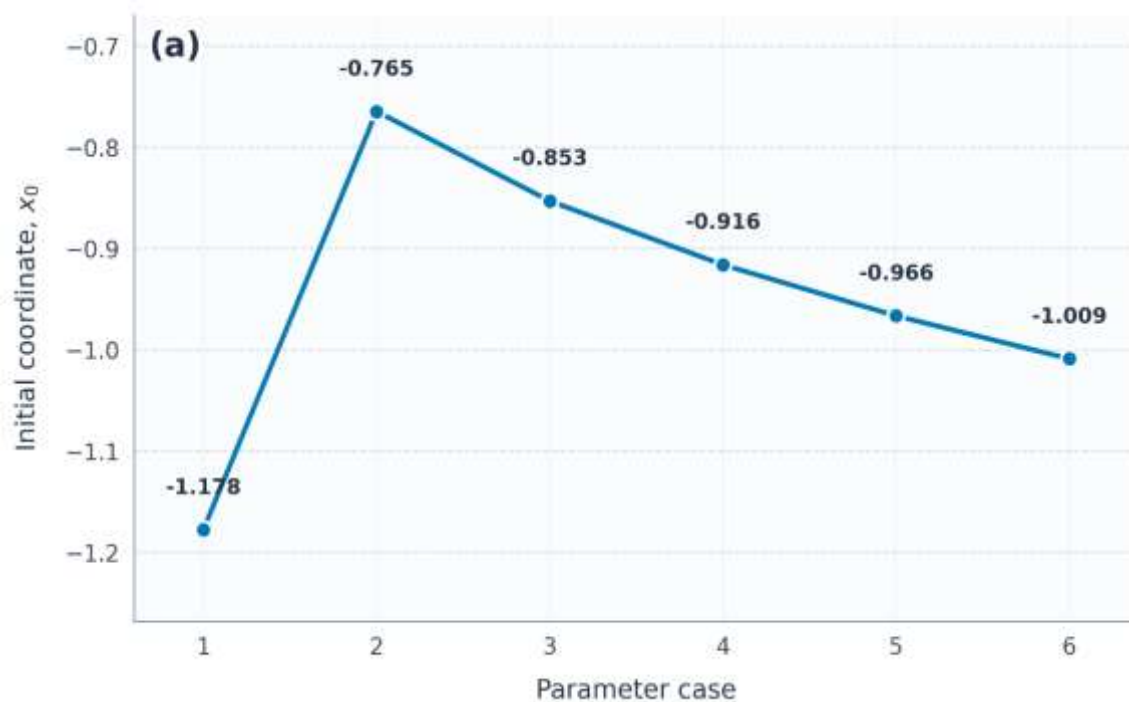


Figure 2(a). Initial x-coordinate near L_1 across parameter Cases 1–6 ($\mu = 0.47333$).

The initial x coordinate varies across the six cases, reaching its least negative value in Case 2 (-0.76451797), compared with -1.17772296 in Case 1. The starting coordinate then shifts to -1.00888380 by Case 6, reflecting differences in the benchmark initial conditions.

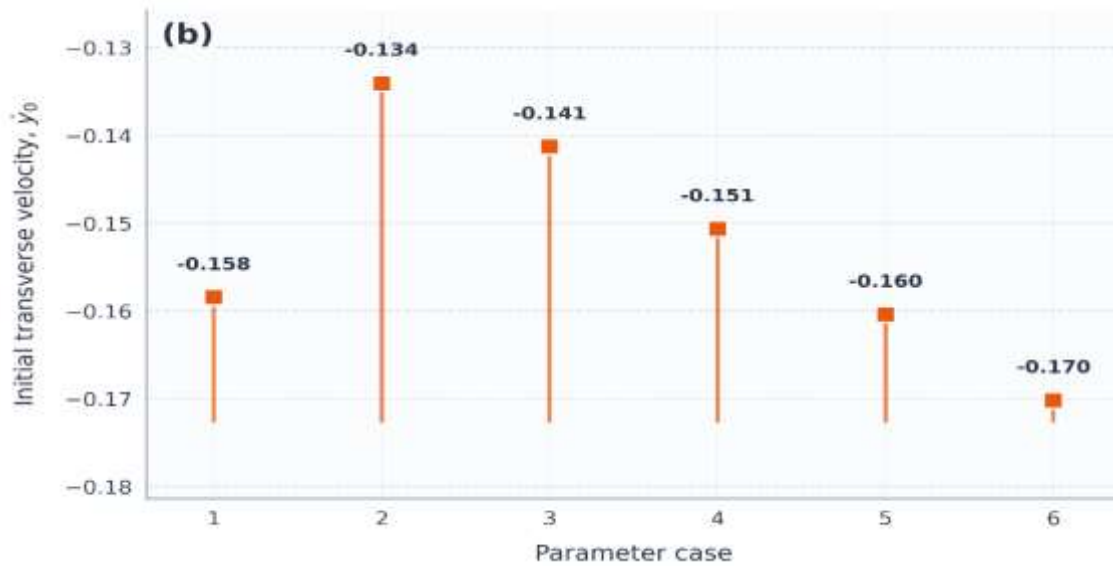


Figure 2(b). Initial transverse velocity near L_1 across parameter Cases 1–6 ($\mu = 0.47333$).

The transverse velocity changes from -0.15844398 in Case 1 to -0.13404904 in Case 2 and subsequently decreases to -0.17018986 in Case 6.

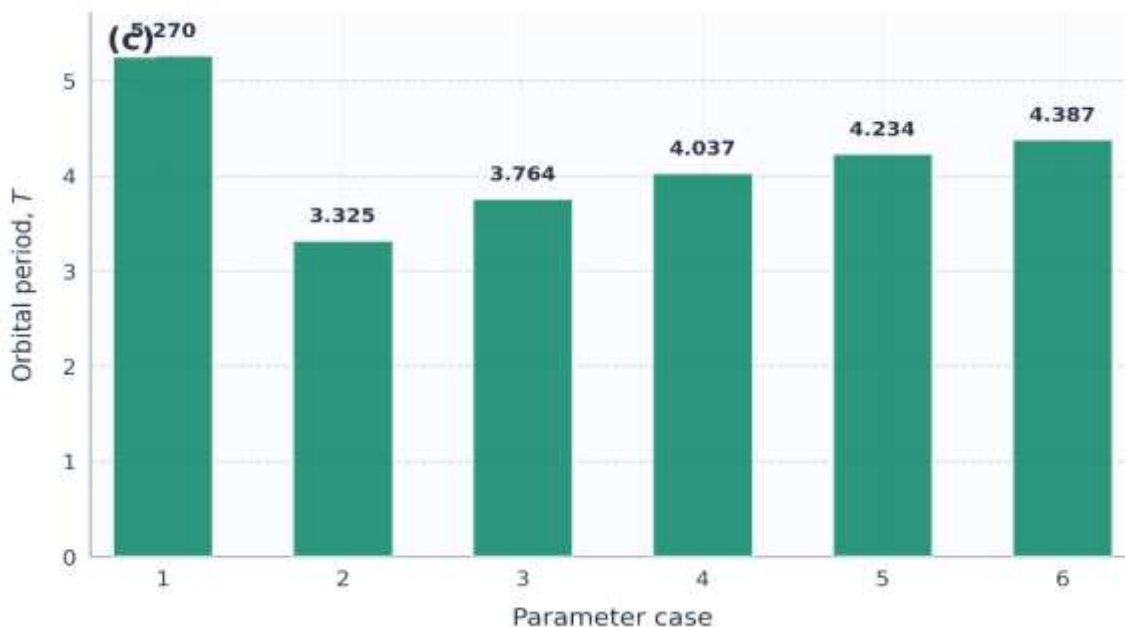


Figure 2(c). Reported orbital period near L_1 across parameter Cases 1–6 ($\mu = 0.47333$).

The period drops from 5.26996513 in Case 1 to 3.32511205 in Case 2, then increases through the remaining cases to 4.38698369 . Since several radiation and oblateness parameters vary simultaneously, the variation should be interpreted as a response across parameter cases rather than an isolated sensitivity.

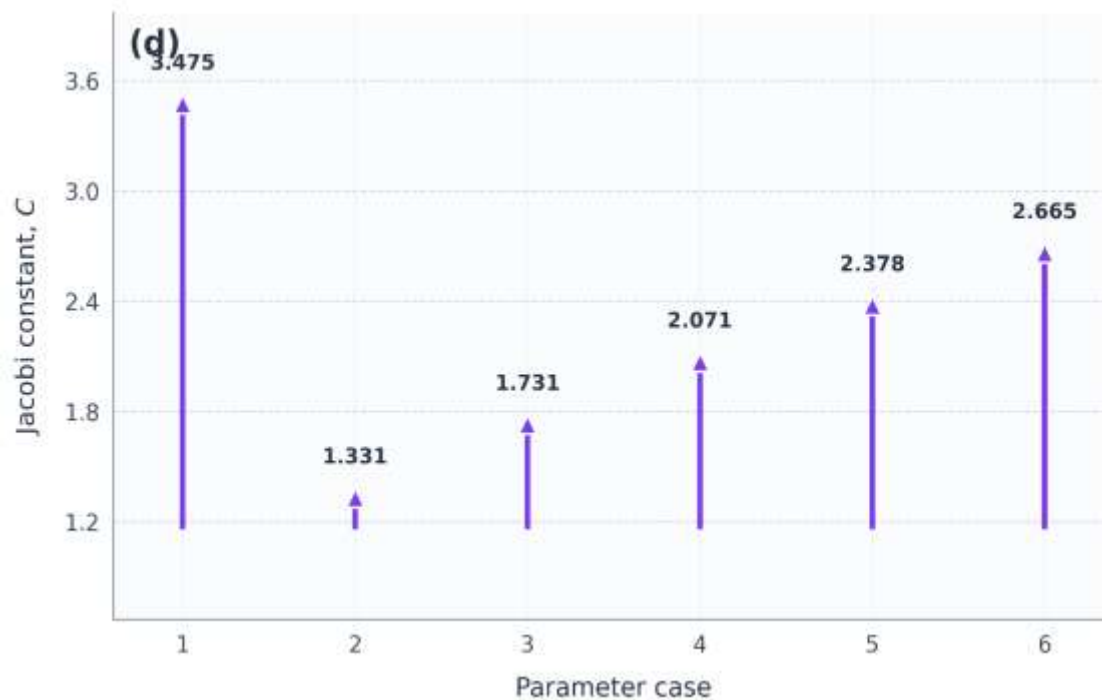


Figure 2(d). Jacobi constant near L_1 across parameter Cases 1–6 ($\mu = 0.47333$). The listed Jacobi constant decreases sharply between Cases 1 and 2, from 3.47518463 to 1.33092657, before increasing to 2.66454003 in Case 6.

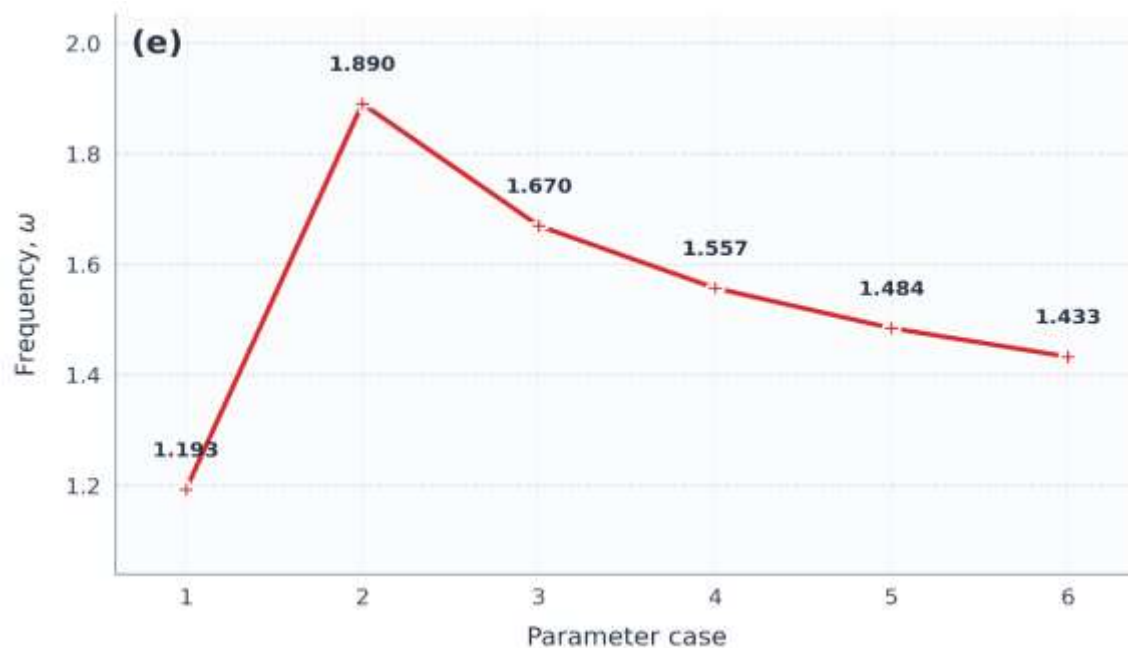


Figure 2(e). The frequency near L_1 across parameter Cases 1–6 ($\mu = 0.47333$).

The frequency increases from 1.19274305 in Case 1 to 1.89037668 in Case 2, then decreases gradually to 1.43281004 in Case 6. These benchmark frequency values may be used for comparison.

Table 3. Starting conditions for Lyapunov orbits near L₂ ($\mu = 0.47333$).

Case	x_0	$\dot{y}_0$	T	C	ω
1	-0.00786810	-0.50240508	1.66199403	3.99865977	3.78203180
2	-0.13576459	-0.09924098	4.49137120	0.95857994	1.39950897
3	-0.10567840	-0.17260044	3.08133233	1.39453273	2.03993390
4	-0.09130772	-0.24469574	2.48534026	1.83036201	2.52911619
5	-0.08353241	-0.31972218	2.12428715	2.27039126	2.95897580
6	-0.07917390	-0.39856305	1.87327991	2.71609925	3.35545918

Note. Reported starting conditions and orbital descriptors;. Numerical values reproduced from Singh and Gyegwe (2017) for comparison.

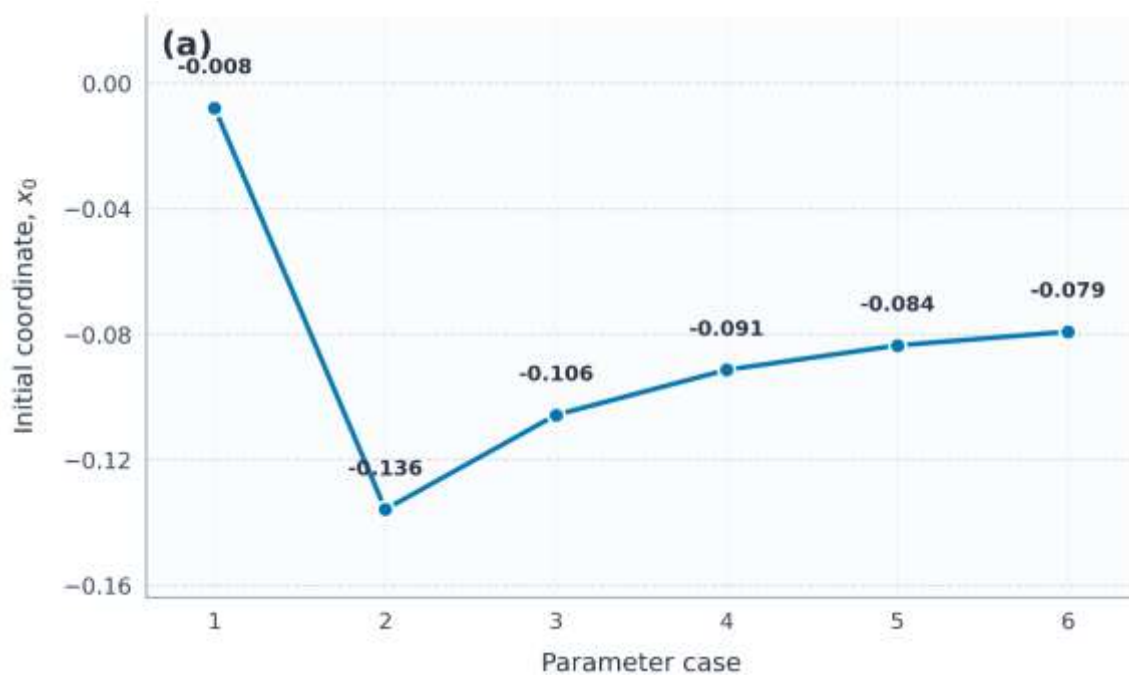


Figure 3(a). Initial x-coordinate near L₂ across parameter Cases 1–6 ($\mu = 0.47333$).

The x-coordinate changes from -0.00786810 in Case 1 to -0.13576459 in Case 2, then becomes less negative across Cases 3–6, reaching -0.07917390 . The observed pattern characterises the benchmark starting positions within the six combined parameter configurations.

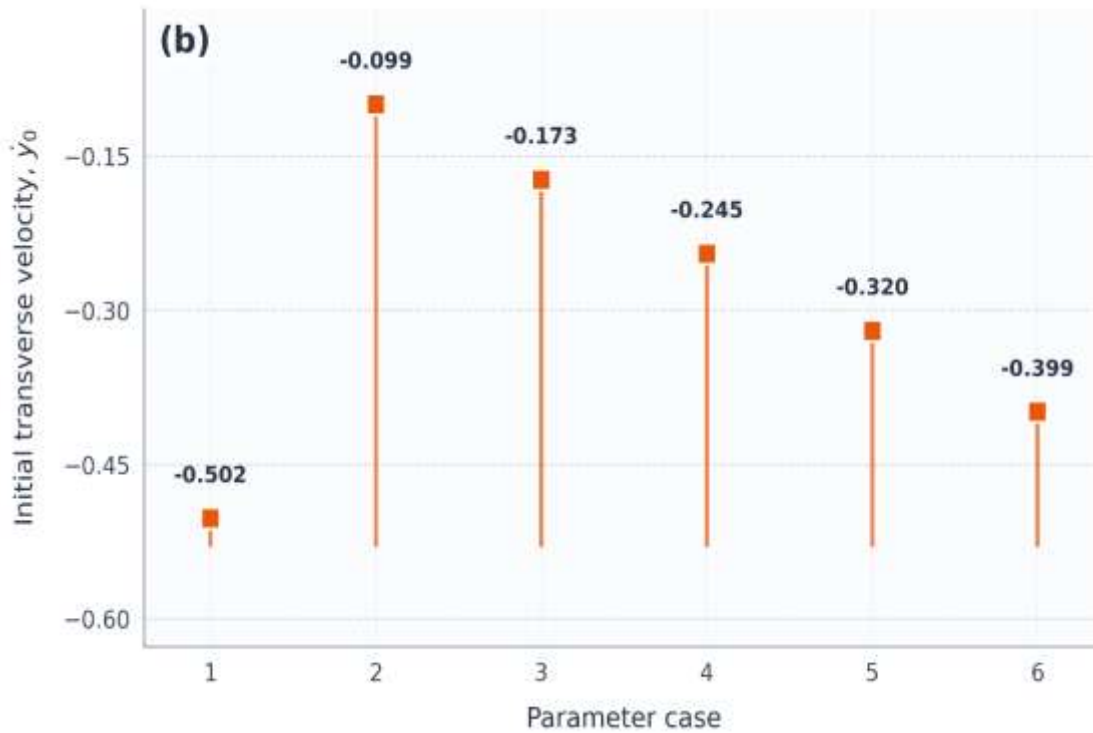


Figure 3(b). Initial transverse velocity near L_2 across parameter Cases 1–6 ($\mu = 0.47333$). The reported transverse velocity is most negative in Case 1 (-0.50240508), rises to -0.09924098 in Case 2, and moves toward -0.39856305 by Case 6.

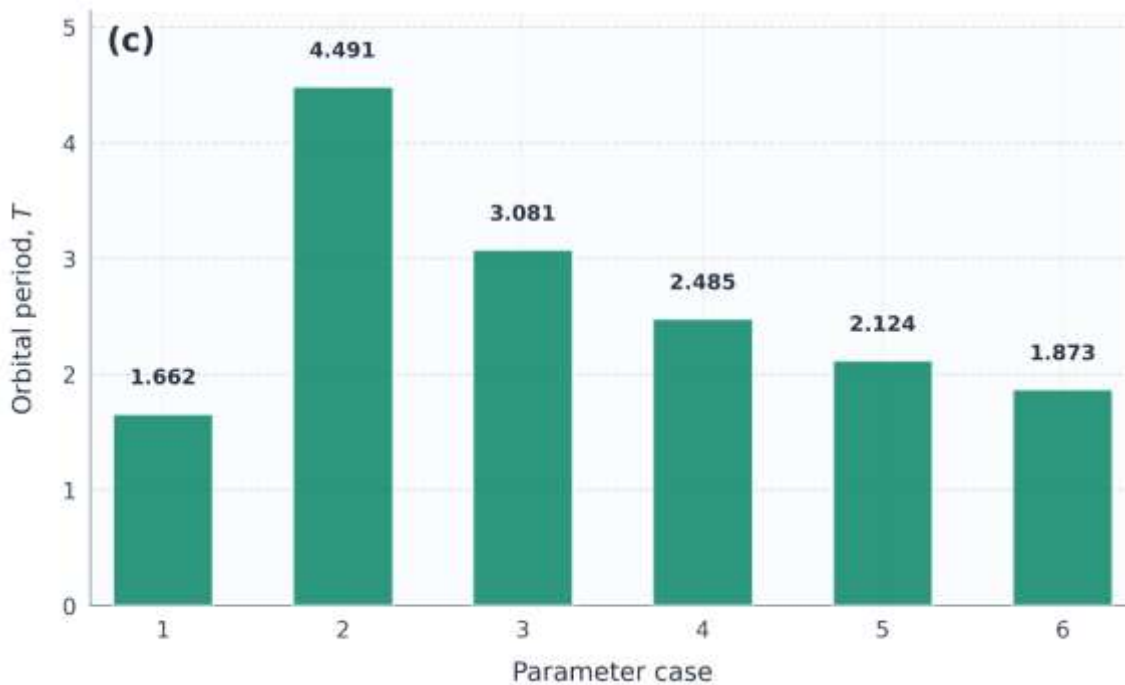


Figure 3(c). The orbital period near L_2 across parameter Cases 1–6 ($\mu = 0.47333$). The period rises from 1.66199403 in Case 1 to 4.49137120 in Case 2 and declines thereafter to 1.87327991 in Case 6.

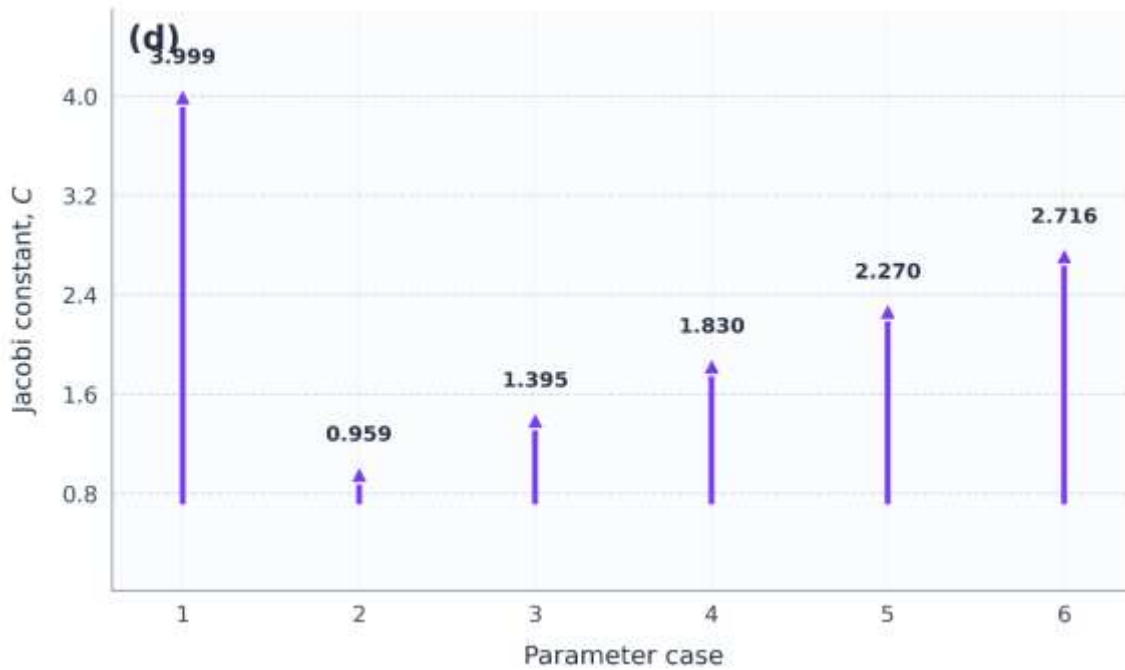


Figure 3(d). Jacobi constant near L_2 across parameter Cases 1–6 ($\mu = 0.47333$). The Jacobi constant is 3.99865977 in Case 1 and 0.95857994 in Case 2, then increases progressively through Case 6 to 2.71609925.

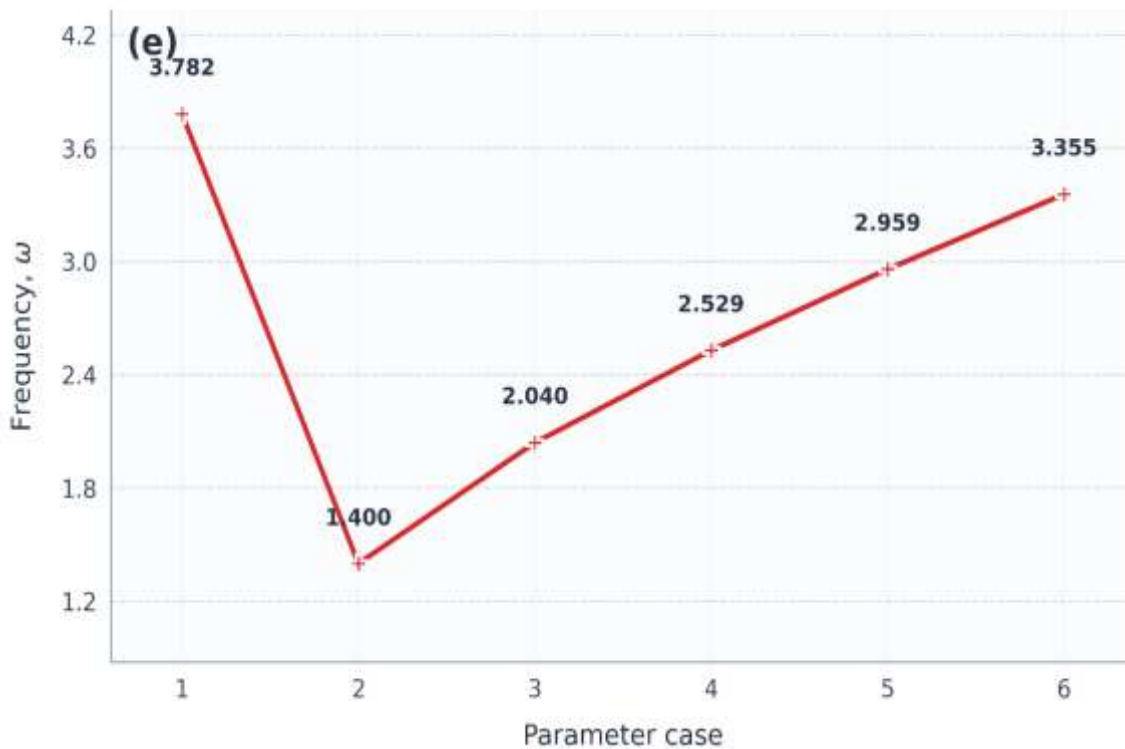


Figure 3(e). Reported frequency near L_2 across parameter Cases 1–6 ($\mu = 0.47333$).

The frequency decreases from 3.78203180 in Case 1 to 1.39950897 in Case 2, then increases to 3.35545918 by Case 6. This marked case dependence provides a useful benchmark pattern for subsequent independent frequency calculations.

Table 4. The starting conditions for Lyapunov orbits near L_3 ($\mu = 0.47333$).

Case	x_0	$\dot{y}_0$	T	C	ω
1	1.21881344	-0.14860729	5.61879746	3.43731738	1.11869387
2	0.98805848	-0.10827148	4.11676337	1.88451257	1.52685829
3	1.04064080	-0.12551658	4.23720387	2.19245704	1.48345807
4	1.08568711	-0.14022498	4.33697474	2.48247540	1.44933154
5	1.12528648	-0.15369264	4.41909062	2.75927019	1.42239995
6	1.16071210	-0.16640484	4.48676943	3.02584346	1.40094435

Note. Reported starting conditions and orbital descriptors; Numerical values from Singh and Gyegwe (2017) for comparison.

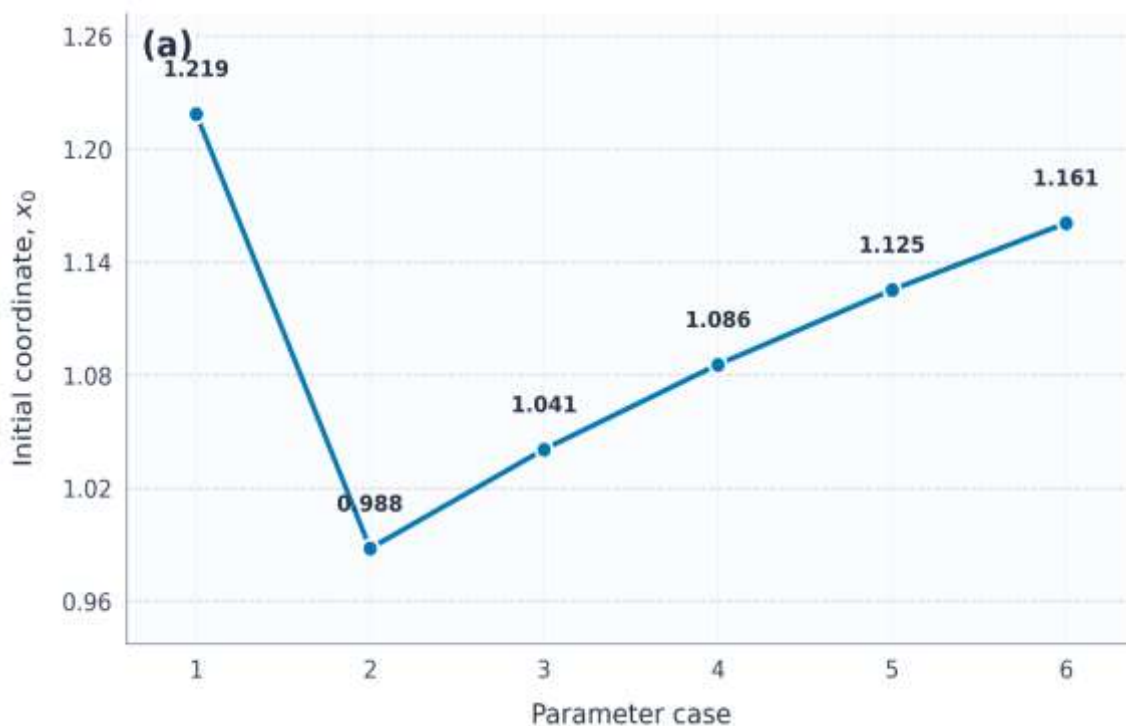


Figure 4(a). Initial x-coordinate near L_3 across parameter Cases 1–6 ($\mu = 0.47333$). The initial coordinate decreases from 1.21881344 in Case 1 to 0.98805848 in Case 2, followed by a progressive increase to 1.16071210 in Case 6. The tabulated movement reflects the case-specific starting positions for the reported L_3 approximations.

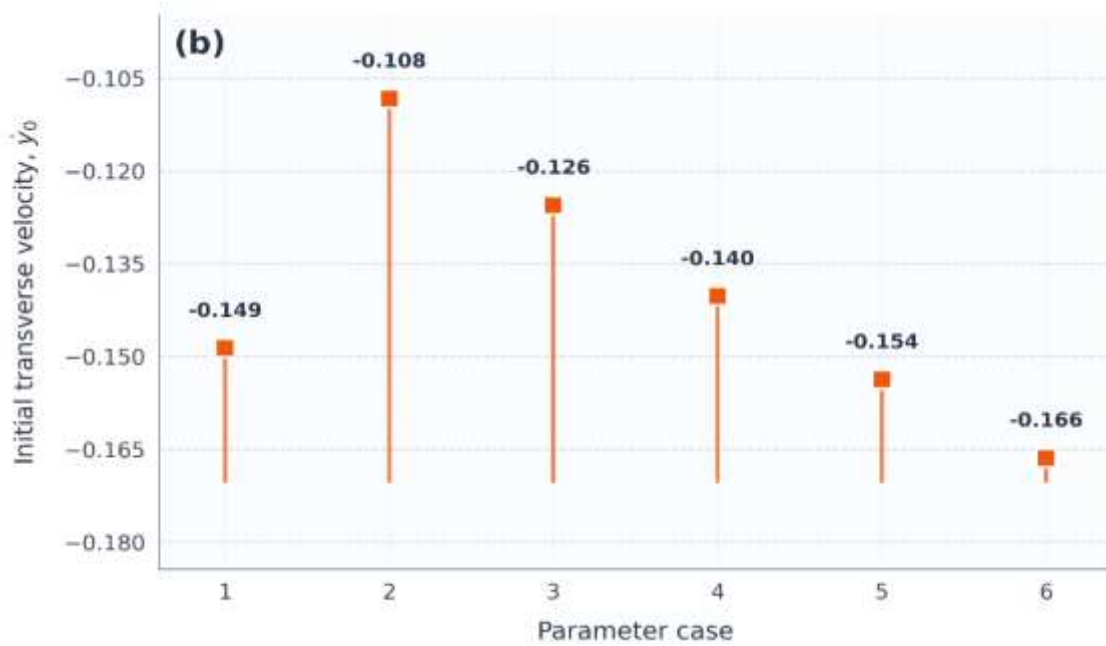


Figure 4(b). Initial transverse velocity near L_3 across parameter Cases 1–6 ($\mu = 0.47333$). The initial transverse velocity changes from -0.14860729 to -0.10827148 between Cases 1 and 2, before becoming more negative in subsequent cases and reaching -0.16640484 in Case 6.

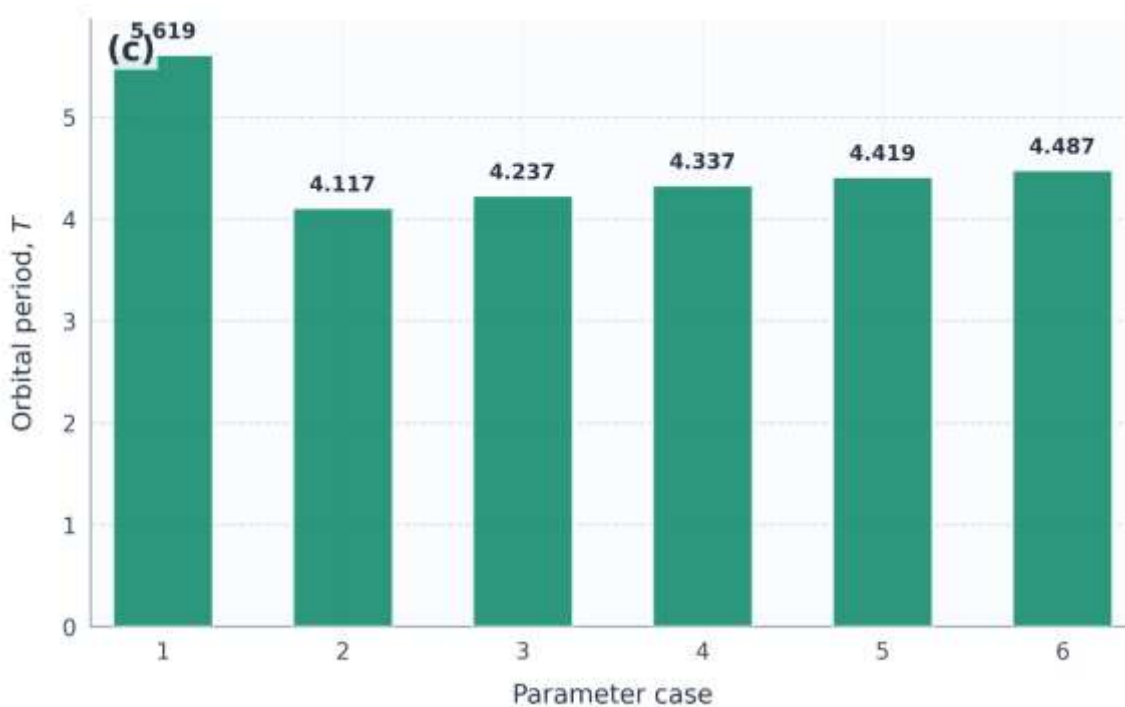


Figure 4(c). The orbital period near L_3 across parameter Cases 1–6 ($\mu = 0.47333$). The reported period decreases from 5.61879746 in Case 1 to 4.11676337 in Case 2, followed by an increase to 4.48676943 in Case 6. Differences among the cases involve simultaneous changes to the radiation and oblateness coefficients.

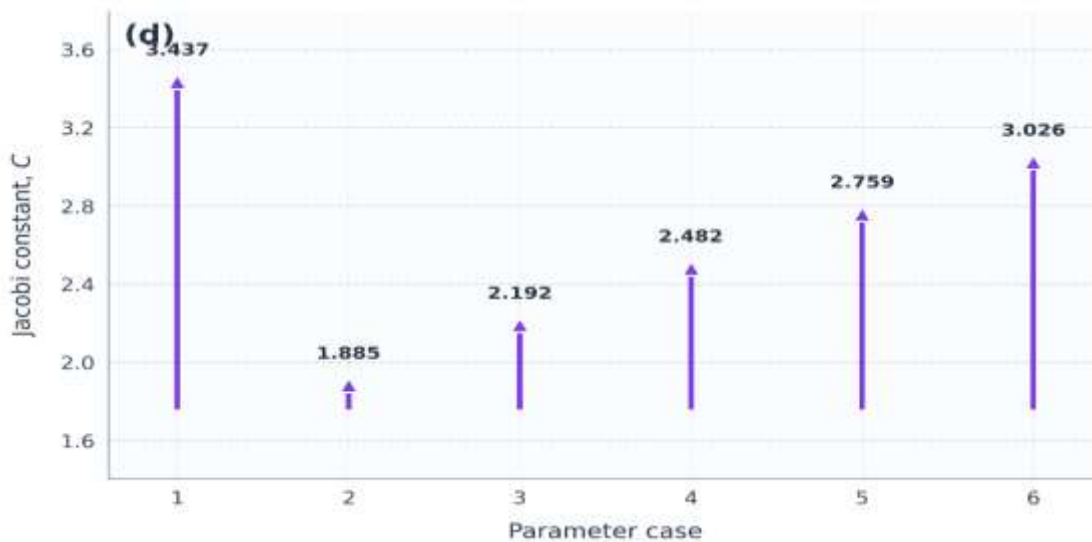


Figure 4(d). Jacobi constant near L_3 across parameter Cases 1–6 ($\mu = 0.47333$). The Jacobi constant decreases from 3.43731738 in Case 1 to 1.88451257 in Case 2 and rises to 3.02584346 by Case 6.

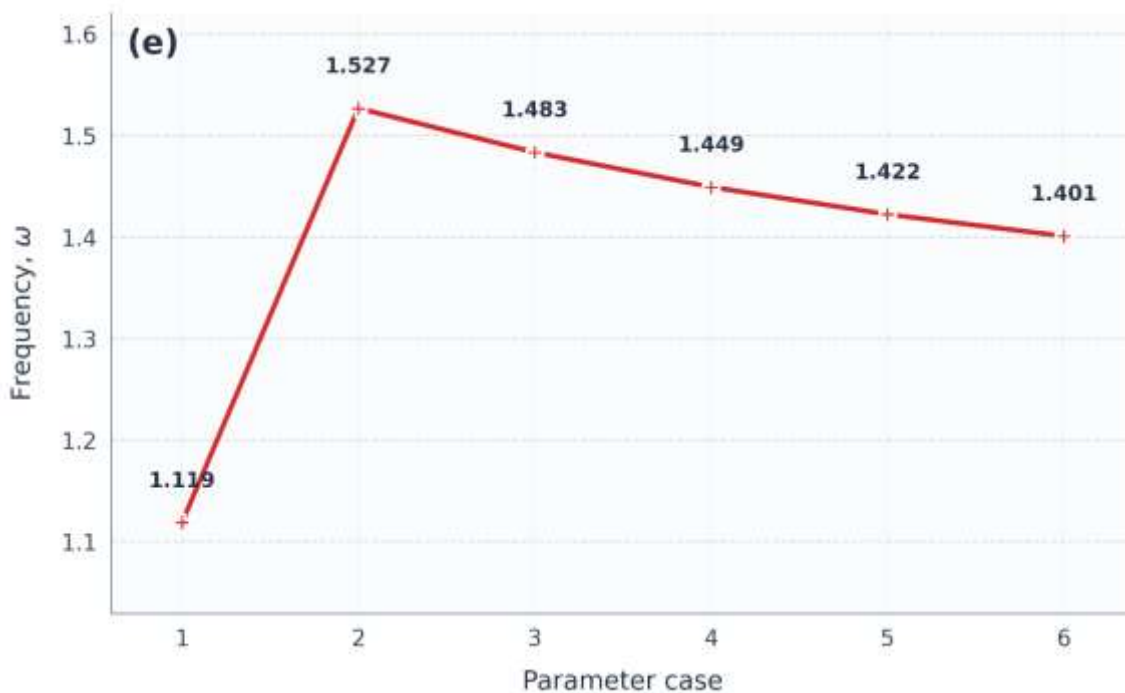


Figure 4(e). Reported frequency near L_3 across parameter Cases 1–6 ($\mu = 0.47333$). The frequency increases from 1.11869387 in Case 1 to 1.52685829 in Case 2 and then declines gradually to 1.40094435 in Case 6.

4. Discussion

The analysis links the equilibrium geometry of the radiating and oblate α -Centauri model to a

systematic local perturbation representation of planar motion. The governing effective potential incorporates the radiation factors q_1 and q_2 and

oblateness coefficients A_1 and A_2 in a common rotating-frame model. Its derivatives determine both the collinear roots and the Taylor coefficients governing local displacement dynamics. Consequently, changes in the equilibrium coordinates also modify the linear oscillatory terms and the nonlinear harmonic contributions. This mathematical connection is central to interpreting the parameter-dependent results rather than treating the plotted coordinates and orbital descriptors as unrelated numerical trends (Abouelmagd, 2013; Celletti et al., 2024).

The six configurations distinguish the unperturbed reference case from five joint radiation–oblateness settings. Between Cases 1 and 2, the reported L_1 coordinate changes from -1.20751483 to -0.79423688 , L_2 from -0.03765997 to -0.16548350 , and L_3 from 1.18902157 to 0.95833957 . Across Cases 2–6, L_1 subsequently moves toward more

negative coordinates, L_2 toward less negative coordinates, and L_3 toward larger positive coordinates. The opposite directional responses of the three roots reflect their different positions relative to the primaries and the distance-dependent terms in the equilibrium equation. Because the four perturbation parameters are varied jointly, these comparisons quantify the response of the specified combined cases, not a separately identifiable effect of either radiation or oblateness. In this respect, the case study complements established investigations of photogravitational equilibria while retaining its

specific α -Centauri parameter convention (Abouelmagd, 2013; Zotos, 2015).

The local Taylor expansion makes the implications of this coordinate response explicit. The first-order coefficients h_1 and D_1 describe the linearised dynamics, whereas the higher-order coefficients introduce the constant, second-harmonic and third-harmonic contributions required for the Lyapunov-type approximation. At cubic order, the Lindstedt–Poincaré rescaling $t = \kappa\tau$, with $\kappa = 1 + \rho\epsilon^2$, links the solvability condition for ρ to the amplitude-dependent period $T = 2\pi\kappa/\omega$. Applying the same transformation to Coriolis coupling and reconstructing velocity through differentiation with respect to physical time ensures that the approximation follows one consistent timescale. The third-order solvability and harmonic-coefficient relations developed in the Word methodology therefore extend the formal description beyond the initial linear oscillation. This analytical structure is comparable in purpose, although not identical in dimension or forcing assumptions, to established perturbative constructions of near-collinear periodic motion (Richardson, 1980; Celletti et al., 2024).

The MATLAB-generated figures provide a consistent visual comparison of the specified numerical tables across all three equilibrium neighbourhoods. In the benchmark data attributed to Singh and Gyegwe (2017), the periods around L_1 change from 5.26996513 in Case 1 to 3.32511205 in Case 2, while those around L_2 change from 1.66199403 to 4.49137120 ; the corresponding L_3 periods change from 5.61879746 to 4.11676337 . These

contrasting responses demonstrate that identical parameter-case changes do not produce identical temporal changes at the three collinear locations. The accompanying listed angular frequencies vary inversely with period and are consistent, to the precision shown, with the benchmark relationship $T \approx 2\pi/\omega$. For the present amplitude-corrected formulation, however, the relevant relation is $T = 2\pi\kappa/\omega$, so quantitative agreement requires the case-specific ε and ρ rather than assuming $\kappa = 1$.

The initial positions and transverse velocities in Tables 2–4 offer a practical comparison basis, but their physical interpretation also depends on model consistency. In particular, the Jacobi constant C must be evaluated from the stated effective potential and the complete initial velocity, and conservation along a computed trajectory is distinct from a comparison of different nominal C values across parameter cases. The reviewer-identified discrepancies in the L_2 Case 1 benchmark C and ω therefore merit explicit computational reconciliation before these entries are used to assert quantitative validation of the present equations. Likewise, the MATLAB plots of tabulated values demonstrate trends but do not themselves constitute independent numerical orbit integrations or closure tests. This distinction strengthens the evidential interpretation of the reported analytical and graphical work without conflating the literature benchmarks with new orbit solutions.

Taken together, the formulation and comparisons establish a coherent route from the perturbed potential through equilibrium determination, local linearisation, harmonic forcing and

amplitude-dependent time scaling to case-specific orbital descriptors. Relative to earlier α -Centauri benchmark tabulations, the present mathematical treatment clarifies how the perturbation equations and the transformed time variable should be assembled when constructing local Lyapunov-type approximations. The combined-case comparison provides useful geometric and temporal context, while independent orbit integration, invariant-drift assessment and direct comparison of newly evaluated coefficients with benchmark values remain the appropriate next tests of predictive accuracy (Howell, 1984; Celletti et al., 2024).

Conclusion

This study formulated the planar radiating and oblate α -Centauri restricted three-body model around its three collinear equilibria and developed the associated local Taylor and Lindstedt–Poincaré relations through cubic order. Its principal analytical contribution lies in connecting the equilibrium-dependent coefficients, the harmonic expansion and the amplitude-dependent time transformation while retaining consistent Coriolis coupling and physical-time velocity reconstruction. Comparisons across six joint parameter configurations reveal distinct directional shifts of the three equilibrium locations and contrasting benchmark patterns in initial conditions, periods, frequencies and Jacobi constants. The MATLAB graphical analysis makes those differences readily comparable within one modelling framework. The cited orbital descriptors remain published benchmarks; case-specific evaluation of the perturbation amplitudes and coefficients,

followed by independent integration and Jacobi-invariant checks, is needed to establish quantitative orbit accuracy. Within this scope, the

formulation supplies a clear mathematical basis for subsequent verification and analysis of near-collinear Lyapunov-type motion.

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