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HOMOCLINIC TRAJECTORIES AND PERIODIC-ORBIT FAMILIES NEAR L_3 IN THE PLANAR PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM

Audu, A. O.,^{1*} Adigun, M. K.², & Dauda, H. A.³

¹Department of Mathematics, Faculty of Science, Federal University of Education, Kontagora, Niger State, Nigeria

²Department of Urban and Regional Planning, School of Environmental Studies, YABATECH, Yaba, Lagos, Nigeria

³Department of Political Science, Faculty of Social Science. University of Lagos

Corresponding author: aaonus@gmail.com

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Abstract

This study examines the structure of non-symmetric homoclinic trajectories and associated symmetric periodic-orbit families near the collinear equilibrium point L_3 in the planar photogravitational restricted three-body problem. The mathematical formulation incorporates radiation-reduction factors into the rotating-frame equations of motion, and fourth-order asymptotic expansions are developed to obtain initial approximations for the outgoing and incoming trajectories. The governing equations, equilibrium conditions and asymptotic solutions provide the mathematical framework for examining doubly asymptotic connections. Numerical integration and iterative phase-space matching are employed to investigate the orbital configurations, while MATLAB software is used to generate graphical representations of the asymptotic trajectories, zero-velocity curves and associated periodic-orbit families. The analysed configurations comprise five orbital classes, designated 1–2, 1–4, 1–6, 1–8 and 1–

10, distinguished by their section-crossing patterns. The results illustrate variations in orbital geometry, equilibrium location and Jacobi constant across the considered mass and radiation parameters. The graphical representations further demonstrate the relationship between symmetry-related homoclinic branches and neighbouring symmetric periodic solutions, whose continued families approach limiting simple periodic orbits of the L_3 Lyapunov family. The study presents a mathematical formulation and graphical analysis of the orbital structures near L_3 , providing a framework for investigating how photogravitational parameters influence homoclinic trajectories and periodic-orbit configurations.

Keywords: Photogravitational restricted three-body problem; homoclinic trajectories; periodic orbits; L_3 equilibrium; numerical continuation; nonlinear dynamics.

1. Introduction

The planar circular restricted three-body problem provides a rotating-frame model for motion of a negligible-mass particle under two

mutually orbiting primaries (Kalantonis et al., 2006). Periodic-orbit families organize much of its nonlinear phase-space structure and supply a framework for comparing different mass ratios (Khan et al., 2026). Classical analysis

established the importance of asymptotic trajectories near collinear libration points (Khan et al., 2026). Stability classification of periodic solutions requires attention to both in-plane and out-of-plane perturbations (Khan et al., 2026). This distinction is reflected in numerical studies of vertical stability along restricted-problem orbit families (Khumla et al., 2026).

Photogravitational dynamics extend the gravitational model by incorporating effective radiation-reduction factors in the attraction terms (Kumar & Kumar, 2026). Radiation pressure is consequently relevant whenever the third body has a sufficiently large area-to-mass ratio (Podlubný, 2025). Radiation forces on small Solar-System particles have also been analyzed beyond purely gravitational motion, including effects that lie outside the conservative model used here (Hayakawa & Hansen, 2023). Earlier investigations have shown that photogravitational perturbations modify the stability properties of triangular equilibria (Przybylska & Maciejewski, 2023; Trinh & Thanh, 2026). Related work examined the behavior of inner collinear equilibria in an elliptic photogravitational setting, which should not be conflated with the circular model of this study (Hristov, 2026; Idrisi et al., 2025).

The construction of asymptotic orbits around collinear points developed from earlier work on symmetric solutions (Khan et al., 2026; Lin et al., 2024). Subsequent numerical investigation extended attention to non-symmetric branches in photogravitational settings (Podlubný, 2025).

Studies of the unrestricted general three-body problem provide a related, but dynamically different, comparison for asymptotic motion (Capiński et al., 2016). Analyses of spiral-type asymptotic solutions clarified the relation between local equilibrium structure and global trajectory geometry (Zambrano-Serrano et al., 2021). The mathematical treatment of saddle-focus homoclinic configurations provides a broader theoretical context for such connections (Kumar et al., 2020). Further investigations of asymptotic-to-periodic transitions show why a global orbit family cannot be inferred from a local eigendirection alone (Trinh & Thanh, 2026). Results for Chermnykh's problem illustrate how modifying the dynamical model changes the location and character of available non-symmetric connections (Ashwin & Bick, 2024). The focal 2006 study identified non-symmetric homoclinic trajectories to L_3 in the planar photogravitational restricted three-body problem and numerically continued neighboring symmetric periodic solutions (Owen & Baresi, 2024). Separate photogravitational computations investigated homoclinic and heteroclinic connections, showing that the global matching problem extends beyond a single orbit class (Alshabanat et al., 2020). Investigations near triangular photogravitational equilibria address a different local geometry and therefore do not establish the L_3 -specific conclusions (Peterson et al., 2024). An oblateness-perturbed restricted problem likewise yielded numerical homoclinic and heteroclinic trajectories, demonstrating

model dependence of the connections (El-Bar & El-Salam, 2018; Gouzi et al., 2025). In the classical planar problem, stable and unstable invariant manifolds provide a geometric interpretation of transitions between neighborhoods of periodic orbits (Liu et al., 2011). Local and global dynamics near collinear equilibria have also been studied through invariant structures and nonlinear continuation (Oliveira, 2021). Analytical series can supply preliminary approximations to periodic motions about collinear points, but these approximations generally require numerical refinement (Calleja et al., 2012). Numerical computation of three-dimensional halo families established a wider orbital context than the planar homoclinic configurations investigated here (Barrabés et al., 2009). Related work mapped the ranges and stability of spatial periodic families near collinear points (Khan et al., 2026). Almost-rectilinear halo orbits represent another spatial family and should not be identified with the planar periodic solutions of the source study (Khan et al., 2026). Semianalytical construction of quasihalo trajectories similarly concerns nearby quasi-periodic motion rather than the particular homoclinic return condition at L_3 (Barcelona et al., 2024). Asymmetric periodic families in the classical restricted problem offer an additional comparison for symmetry breaking and continuation (Koon et al., 2000; Rosales et al., 2023).

Radiation and oblateness can alter the period and geometry of periodic solutions, although results obtained around triangular points cannot be transferred directly to L_3 (Kaur et al., 2024; Khan et al., 2026). Computational optimization has been used to locate members of periodic families, supporting the broader need for accurate numerical correction (Ma et al., 2021). Studies of spatial photogravitational orbit families with oblateness further emphasize the sensitivity of continuation results to added forces and dimensionality (Kumar et al., 2025). More recent analysis of stable halo-orbit neighborhoods illustrates the ongoing role of invariant structures in characterizing nearby trajectories while remaining outside the planar radiation-only formulation (Acciarini et al., 2024).

Methodology

2.1 Governing dynamical equations and coordinate transformation

A dimensionless barycentric rotating frame is adopted for the planar photogravitational restricted three-body problem. The two primaries have normalized masses $1 - m$ and m , where $0 < m \leq 1/2$, and radiation reduction factors p_1 and p_2 . The particle coordinates are (X, Y) , with distances R_1 and R_2 from the primaries. The equations of motion, are.

$$\begin{aligned}\ddot{X} - 2\dot{Y} &= X - \frac{p_1(1-m)(X+m)}{R_1^3} - \frac{p_2m(X+m-1)}{R_2^3}, \\ \ddot{Y} + 2\dot{X} &= Y \left(1 - \frac{p_1(1-m)}{R_1^3} - \frac{p_2m}{R_2^3} \right).\end{aligned}\quad (1)$$

$$R_1 = \sqrt{(X+m)^2 + Y^2}, \quad R_2 = \sqrt{(X+m-1)^2 + Y^2}. \quad (2)$$

The Jacobi integral is expressed in the sign convention of the article as

$$C = \dot{X}^2 + \dot{Y}^2 - X^2 - Y^2 - \frac{2p_1(1-m)}{R_1} - \frac{2p_2m}{R_2}. \quad (3)$$

The abscissa X_{L_k} of a collinear equilibrium L_k ($k = 1, 2, 3$) satisfies

$$X_{L_k} - \frac{p_1(1-m)(X_{L_k}+m)}{|X_{L_k}+m|^3} - \frac{p_2m(X_{L_k}+m-1)}{|X_{L_k}+m-1|^3} = 0. \quad (4)$$

Introduce local variables $u = X - X_{L_k}$ and $v = Y$. In these variables the equations become

$$\begin{aligned}\ddot{u} - 2\dot{v} &= X_{L_k} + u - \frac{p_1(1-m)(X_{L_k}+u+m)}{\mathcal{R}_1^3} - \frac{p_2m(X_{L_k}+u+m-1)}{\mathcal{R}_2^3}, \\ \ddot{v} + 2\dot{u} &= v \left(1 - \frac{p_1(1-m)}{\mathcal{R}_1^3} - \frac{p_2m}{\mathcal{R}_2^3} \right),\end{aligned}\quad (5)$$

where $\mathcal{R}_1 = \sqrt{(X_{L_k}+u+m)^2 + v^2}$ and $\mathcal{R}_2 = \sqrt{(X_{L_k}+u+m-1)^2 + v^2}$.

2.2 Fourth-order local expansion

Expanding the equations through total polynomial degree four, the local system is

$$\begin{aligned}\ddot{u} - 2\dot{v} &= A_1u + A_2u^2 + A_3v^2 + A_4uv^2 + A_5u^3 + A_6u^4 + A_7v^4 + A_8u^2v^2, \\ \ddot{v} + 2\dot{u} &= B_1v + B_2uv + B_3u^2v + B_4v^3 + B_5uv^3 + B_6u^3v.\end{aligned}\quad (6)$$

The coefficients $A_1, \dots, A_8$ and $B_1, \dots, B_6$ depend on the equilibrium position and the physical parameters. The linear part admits an exponential-mode representation:

$$u(t) = \sum_{k=1}^4 a_k e^{\lambda_k t}, \quad v(t) = \sum_{k=1}^4 b_k e^{\lambda_k t}. \quad (7)$$

Here λ_k are the characteristic exponents of the linearized system, and a_k, b_k denote modal amplitudes. The local orbit construction uses a positive real exponent $\lambda > 0$ for the outgoing branch.

2.3 Outgoing and incoming asymptotic branches

Using the small orbit parameter δ , the fourth-order outgoing asymptotic series is

$$\begin{aligned} u_{\text{out}}(t) &= \sum_{k=1}^4 h_k (\pm\delta)^k e^{k\lambda t}, & v_{\text{out}}(t) &= \sum_{k=1}^4 g_k (\pm\delta)^k e^{k\lambda t}, \\ \dot{u}_{\text{out}}(t) &= \lambda \sum_{k=1}^4 k h_k (\pm\delta)^k e^{k\lambda t}, & \dot{v}_{\text{out}}(t) &= \lambda \sum_{k=1}^4 k g_k (\pm\delta)^k e^{k\lambda t}. \end{aligned} \quad (8)$$

The positive and negative choices define outgoing types I and II, respectively. The symmetry-related incoming branches are

$$\begin{aligned} u_{\text{in}}(t) &= \sum_{k=1}^4 h_k (\pm\delta)^k e^{-k\lambda t}, & v_{\text{in}}(t) &= -\sum_{k=1}^4 g_k (\pm\delta)^k e^{-k\lambda t}, \\ \dot{u}_{\text{in}}(t) &= -\lambda \sum_{k=1}^4 k h_k (\pm\delta)^k e^{-k\lambda t}, & \dot{v}_{\text{in}}(t) &= \lambda \sum_{k=1}^4 k g_k (\pm\delta)^k e^{-k\lambda t}. \end{aligned} \quad (9)$$

The positive and negative choices define incoming types III and IV. At $t = 0$, the four initial-state vectors can be formed as:

$$\begin{aligned} X_{0,\text{I}} &= X_{L_k} + \sum_{j=1}^4 h_j \delta^j, & Y_{0,\text{I}} &= \sum_{j=1}^4 g_j \delta^j, \\ \dot{X}_{0,\text{I}} &= \lambda \sum_{j=1}^4 j h_j \delta^j, & \dot{Y}_{0,\text{I}} &= \lambda \sum_{j=1}^4 j g_j \delta^j. \end{aligned} \quad (10)$$

$$\begin{aligned} X_{0,\text{II}} &= X_{L_k} + \sum_{j=1}^4 h_j (-\delta)^j, & Y_{0,\text{II}} &= \sum_{j=1}^4 g_j (-\delta)^j, \\ \dot{X}_{0,\text{II}} &= \lambda \sum_{j=1}^4 j h_j (-\delta)^j, & \dot{Y}_{0,\text{II}} &= \lambda \sum_{j=1}^4 j g_j (-\delta)^j. \end{aligned} \quad (11)$$

$$\begin{aligned}
X_{0,\text{III}} &= X_{L_k} + \sum_{j=1}^4 h_j \delta^j, & Y_{0,\text{III}} &= -\sum_{j=1}^4 g_j \delta^j, \\
\dot{X}_{0,\text{III}} &= -\lambda \sum_{j=1}^4 j h_j \delta^j, & \dot{Y}_{0,\text{III}} &= \lambda \sum_{j=1}^4 j g_j \delta^j.
\end{aligned} \tag{12}$$

$$\begin{aligned}
X_{0,\text{IV}} &= X_{L_k} + \sum_{j=1}^4 h_j (-\delta)^j, & Y_{0,\text{IV}} &= -\sum_{j=1}^4 g_j (-\delta)^j, \\
\dot{X}_{0,\text{IV}} &= -\lambda \sum_{j=1}^4 j h_j (-\delta)^j, & \dot{Y}_{0,\text{IV}} &= \lambda \sum_{j=1}^4 j g_j (-\delta)^j.
\end{aligned} \tag{13}$$

2.4 Homoclinic matching and numerical correction

The numerical investigation fixes $p_2 = 1$, so only the primary of mass $1 - m$ is luminous. Outgoing type I and incoming type IV branches (or their symmetry-related type II and III branches) are integrated in opposite time directions. Matching a type I and type IV pair requires equality of the two position and velocity states:

$$X_I = X_{\text{IV}}, \quad Y_I = Y_{\text{IV}}, \quad \dot{X}_I = \dot{X}_{\text{IV}}, \quad \dot{Y}_I = \dot{Y}_{\text{IV}}. \tag{14}$$

The procedure imposes the matching at the Y -axis, where $X_I = X_{\text{IV}} = 0$. The remaining two scalar correction equations are

$$Y_I(m, p_1) - Y_{\text{IV}}(m, p_1) = 0, \quad \dot{Y}_I(m, p_1) - \dot{Y}_{\text{IV}}(m, p_1) = 0. \tag{15}$$

A grid search over (m, p_1) provides candidate orbit pairs. Numerical forward integration of the outgoing branch is stopped at its first crossing of the Y -axis; backward integration of the incoming branch is stopped at an even-numbered crossing. Candidates failing the initial mismatch tolerance $\varepsilon_m \leq 10^{-2}$ are discarded. The remaining parameter pairs are corrected by linearizing the two matching equations and repeating the integrations until the stated strict tolerance $\varepsilon_s \leq 10^{-8}$ is reached. The orbital parameter is $\delta = -0.00001$ for the 1–2 orbit class in Table 1 and $\delta = -0.001$ for the 1–4, 1–6, 1–8, and 1–10 classes in Tables 2–5.

2.5 Periodic solutions and continuation

For every matched I–IV homoclinic orbit and its symmetric II–III counterpart, a neighbouring symmetric periodic orbit is initialized by imposing $Y_0 = \dot{X}_0 = 0$ on the homoclinic initial-state approximation. Numerical continuation then follows the associated periodic family. The family

coordinates include X_0 , $\dot{Y}_0$, the half-period $T/2$, the crossing position $X(T/2)$, and the Jacobi integral C . The article reports an isoenergetic correction to identify limiting simple periodic orbits belonging to the Lyapunov family originating at L_3 . The original numerical configurations, half-periods, orbit values, and class labels are retained in the published tables and are not altered in the present mathematical transcription.

3. Numerical Results

3.1 Asymptotic orbit configurations and collinear geometry

The configurations use the rotating-frame variables (X, Y) , the mass parameter m , and radiation factors p_1 and p_2 defined in the methodology. For the numerical orbit survey, $p_2 = 1$. The four asymptotic branches I–IV are distinguished by their series initial conditions and time direction. The illustrative configuration for the four branches and the zero-velocity geometry has $m = 0.1$, $p_1 = 0.5$, $p_2 = 1$, with $\delta = 0.005$ for the illustrated asymptotic branches (Kalantonis et al., 2006).

Figure 1. The four possible types of asymptotic orbits to the collinear equilibrium point L_3 for $m = 0.1$, $p_1 = 0.5$, $p_2 = 1$ and $\delta = 0.005$.

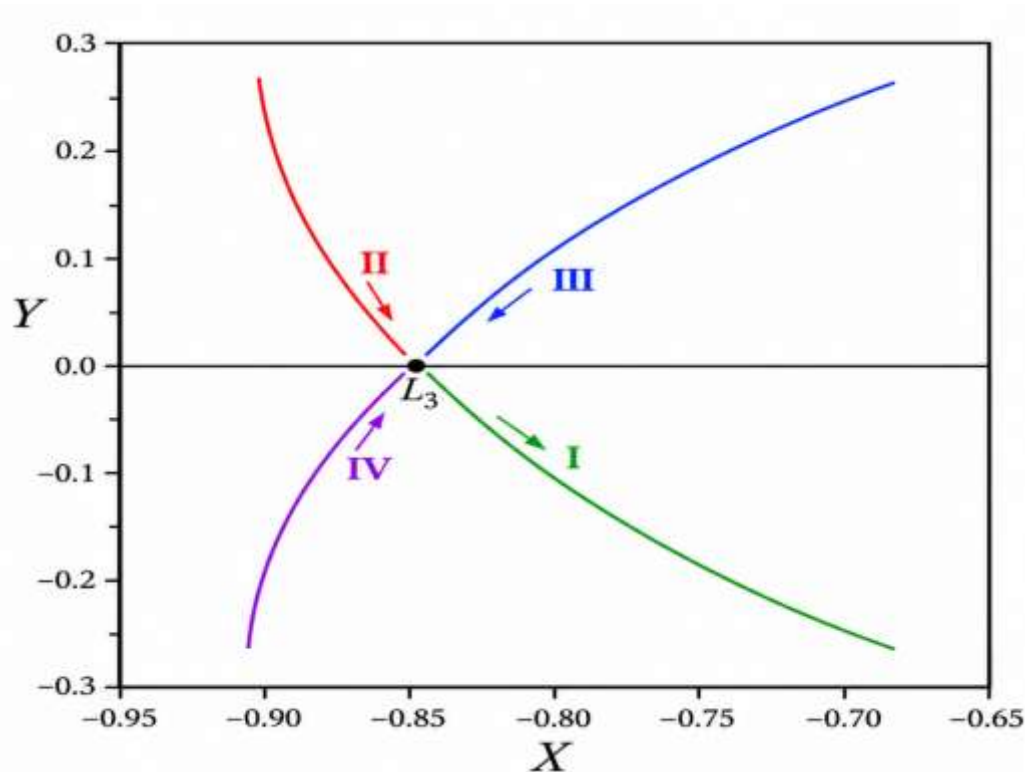
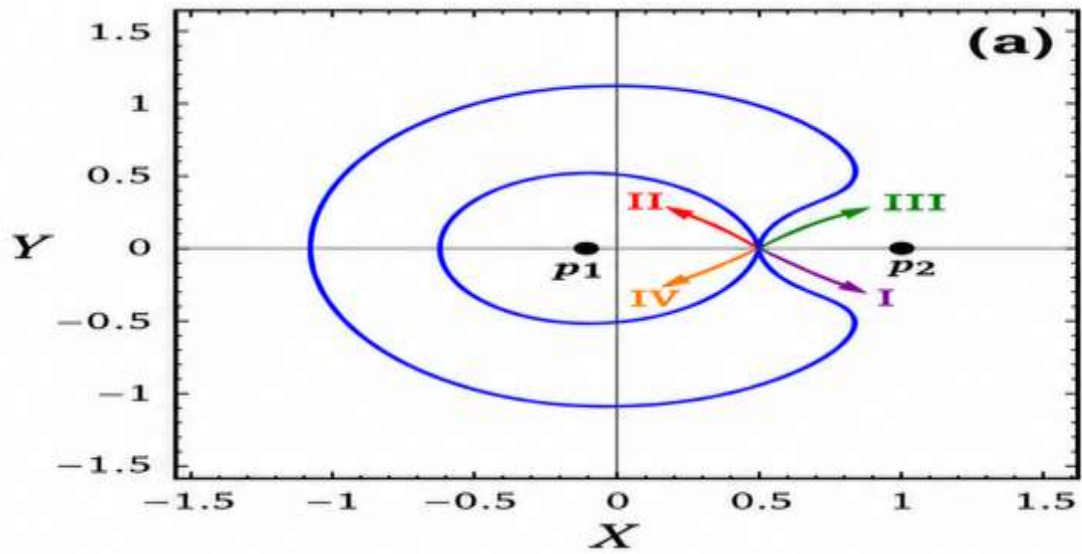
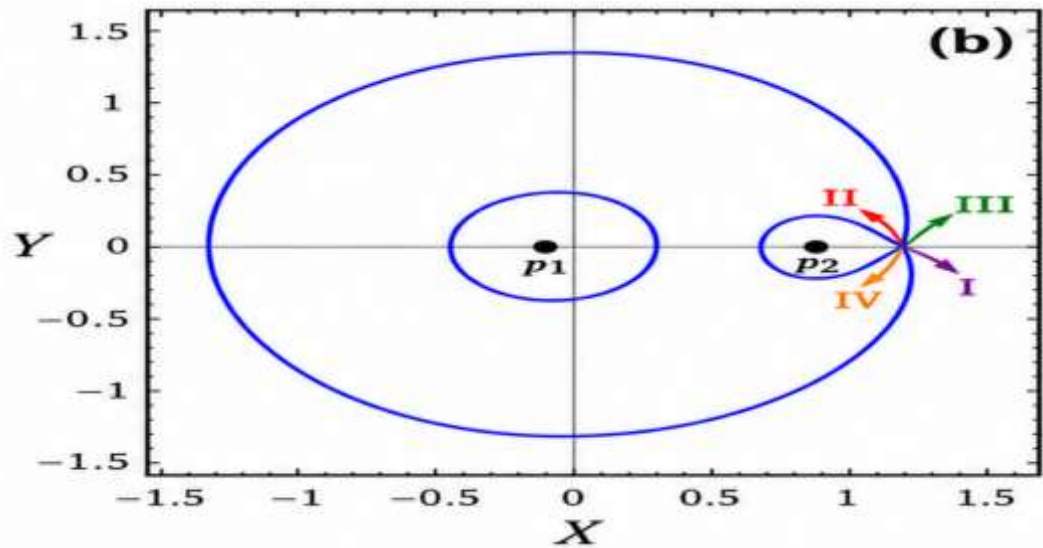
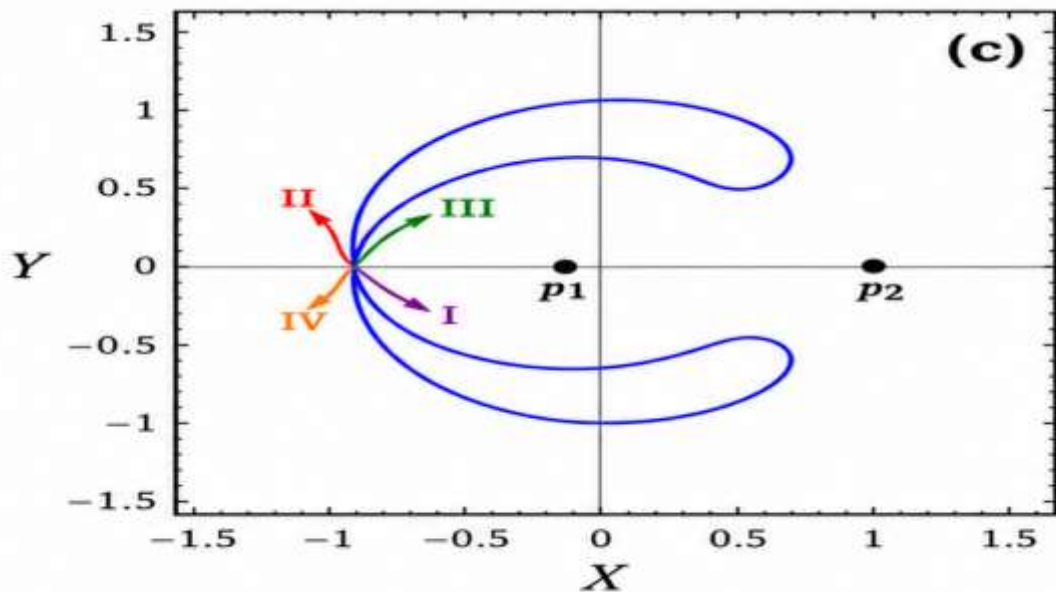


Figure 2. The zero-velocity curves together with the types of asymptotic orbits to the collinear equilibria for $m = 0.1$, $p_1 = 0.5$ and $p_2 = 1$: (a) L_1 , (b) L_2 and (c) L_3 .



For the identified doubly asymptotic solutions, the tables stated the orbit index, the two varied parameters m and p_1 , the L_3 abscissa $X(L_3)$, and the Jacobi constant C . Table 1 uses $\delta = -0.00001$; Tables 2–5 use $\delta = -0.001$. The five orbit classes distinguish one outgoing Y-axis crossing from two, four, six, eight or ten incoming crossings.

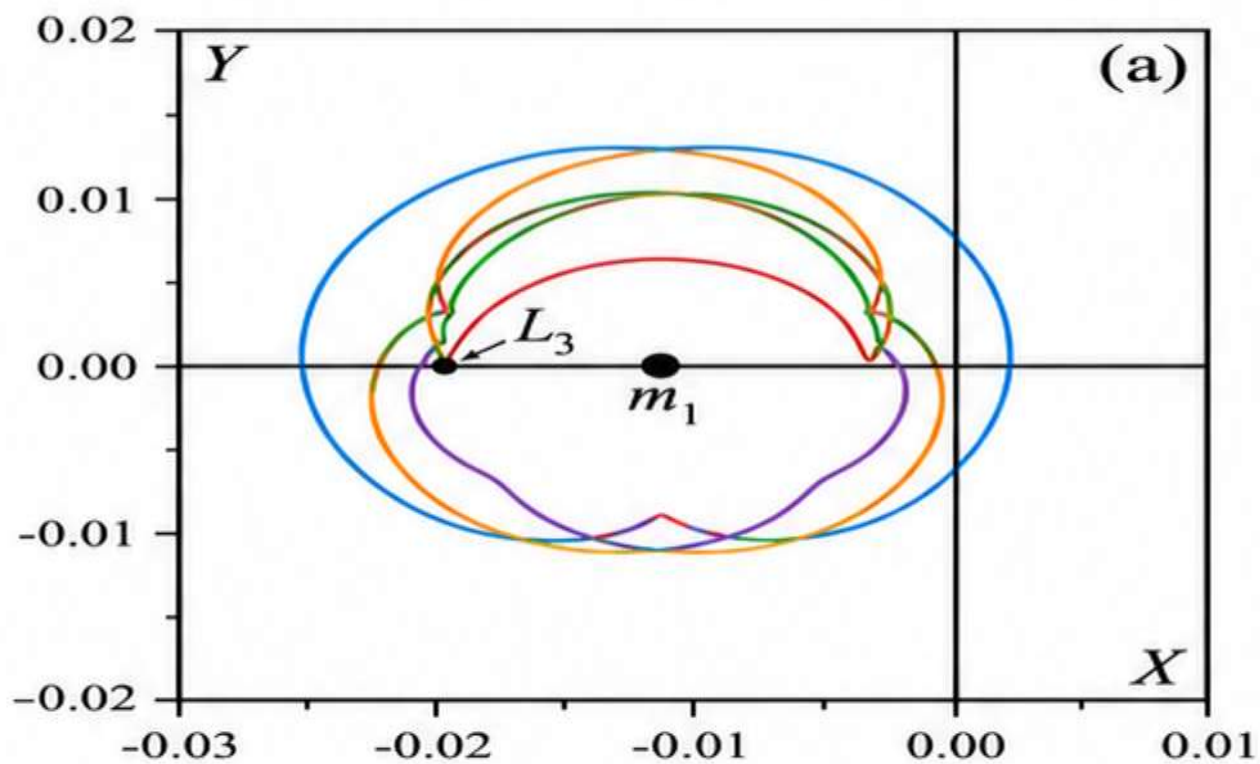
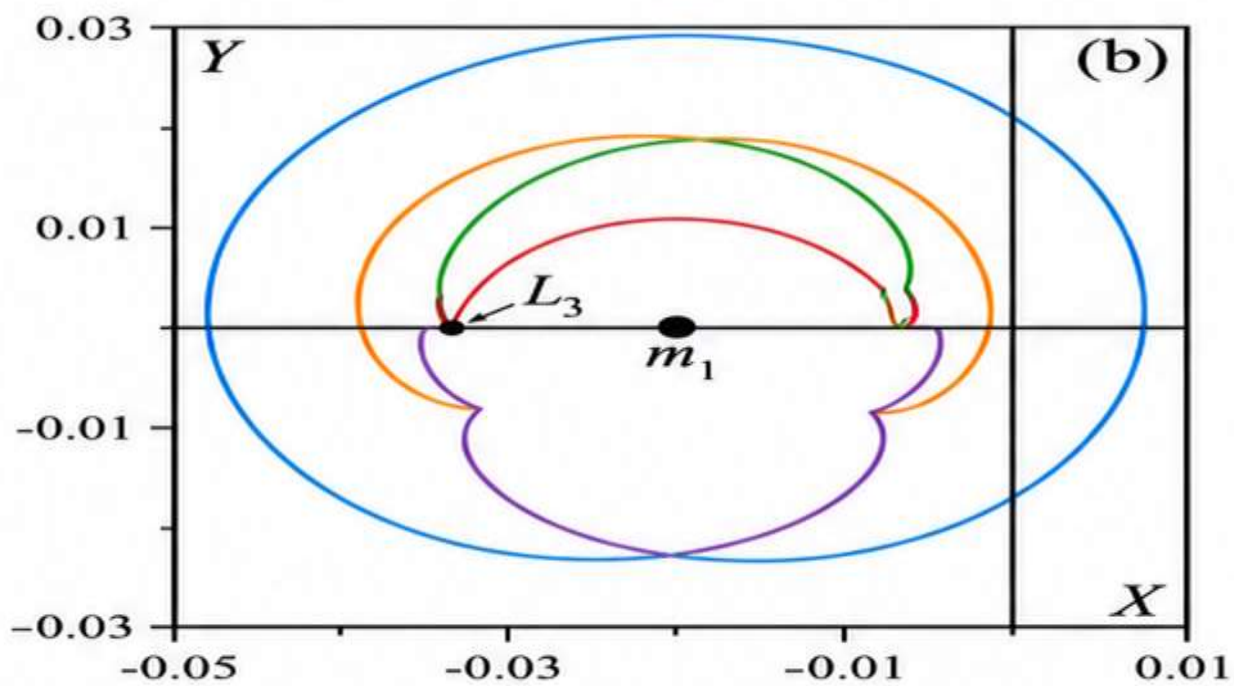
Table 1: Non-symmetric doubly asymptotic 1–2 orbits ($\delta = -0.00001$)

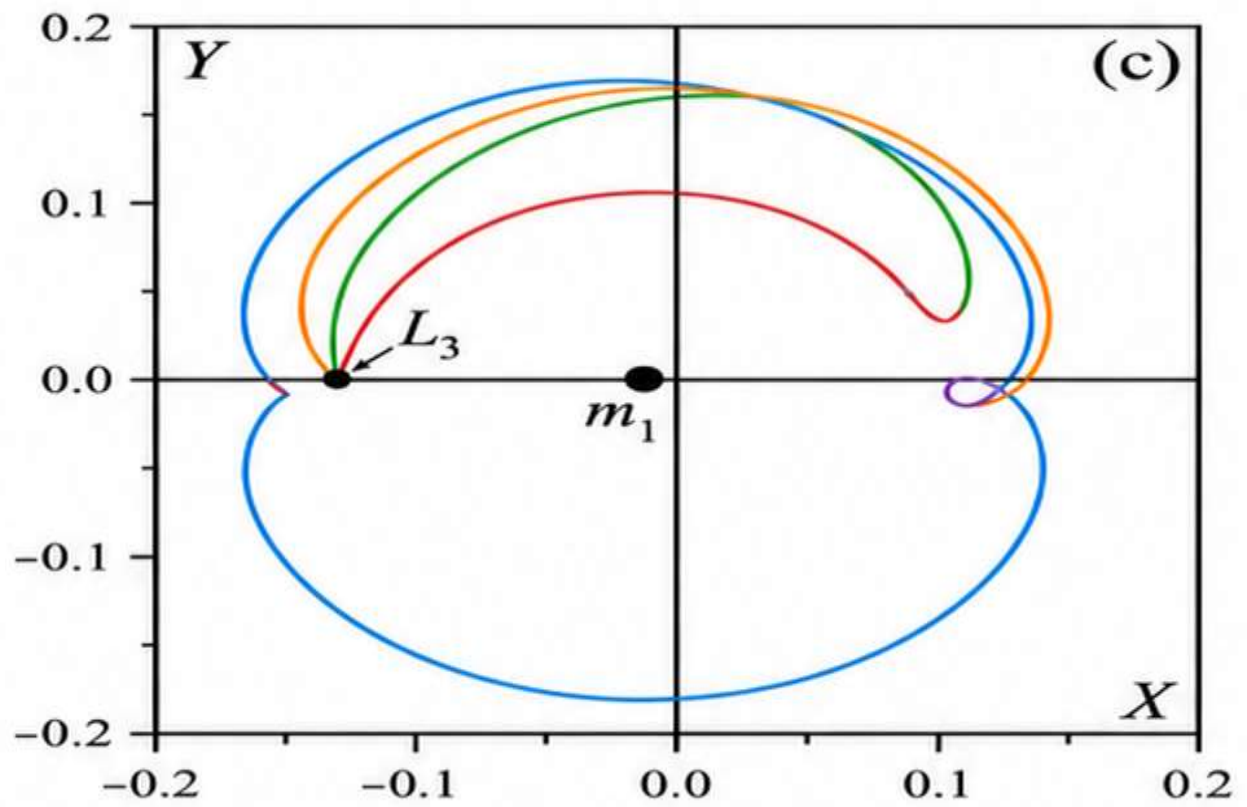
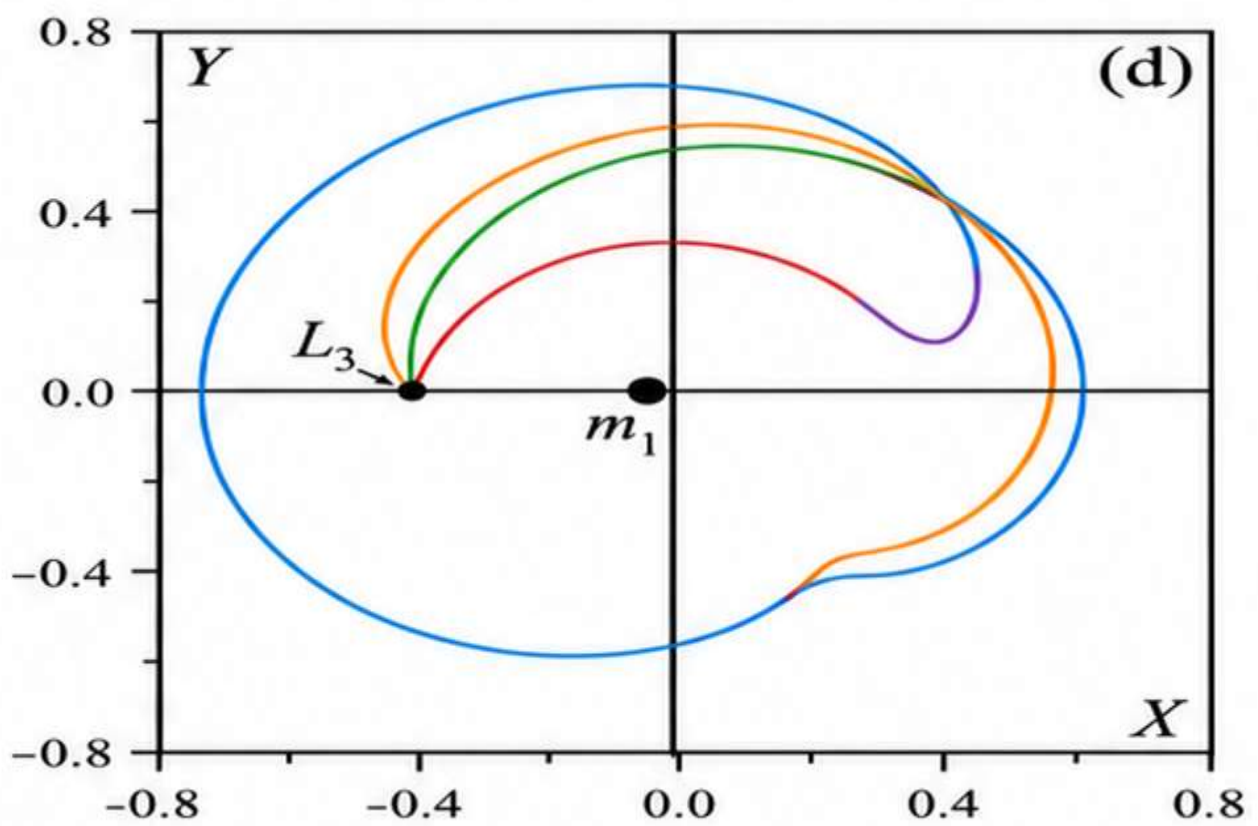
Orbit	m	p_1	$X(L_3)$	C
1	0.0115227	0.0000005	−0.0194535	−0.0233711
2	0.0118889	0.0000005	−0.0197853	−0.0241106
3	0.0198384	0.0000023	−0.0328518	−0.0405982

Table 2: Non-symmetric doubly asymptotic 1–4 orbits ($\delta = -0.001$)

Orbit	m	p_1	$X(L_3)$	C
1	0.0143099	0.0016628	−0.1312751	−0.0708821
2	0.0151113	0.0002384	−0.0762446	−0.0419760
3	0.0157067	0.0016546	−0.1323322	−0.0735726
4	0.0173301	0.0230118	−0.2978767	−0.2770041
5	0.0180790	0.0497895	−0.3809730	−0.4411123
6	0.0185784	0.1971685	−0.5933101	−1.0489890
7	0.0186689	0.1651097	−0.5602549	−0.9364494
8	0.0188611	0.1904354	−0.5868440	−1.0263636
9	0.0262222	0.0594743	−0.4089954	−0.5078104
10	0.0264418	0.0004912	−0.1034459	−0.0722243
11	0.0273991	0.0075578	−0.2191693	−0.1706772
12	0.2827127	0.3974554	−0.8879594	−2.0827689
13	0.0402485	0.0337993	−0.3536335	−0.3933690
14	0.0411867	0.0065564	−0.2219439	−0.1885785
15	0.0417171	0.0008182	−0.1317766	−0.1113178

Figure 3. Non-symmetric doubly asymptotic orbits to L_3 : (a) and (b) show 1–2 orbits (entries 1 and 3 of Table 1); (c)–(e) show 1–4 orbits (entries 3, 9 and 12 of Table 2).





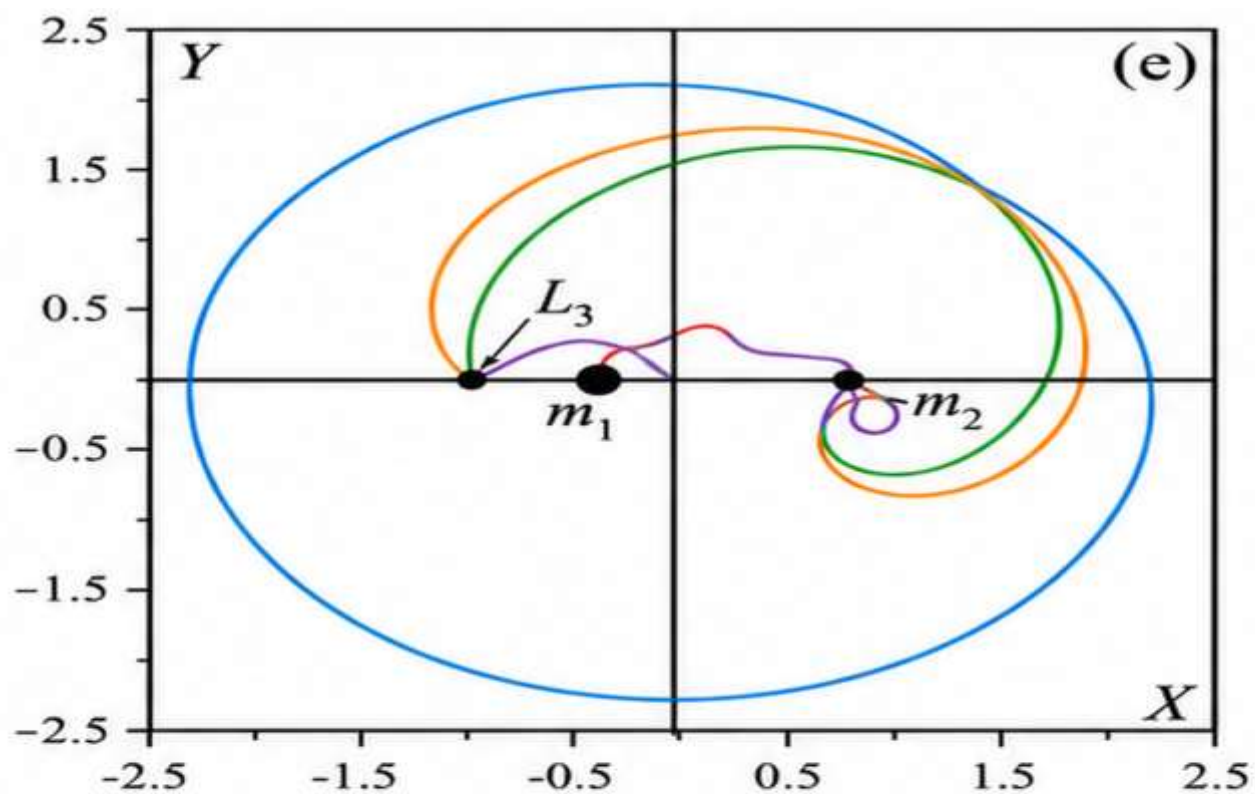


Table 3: Non-symmetric doubly asymptotic 1–6 orbits ($\delta = -0.001$)

Orbit	m	p_1	$X(L_3)$	C
1	0.0107001	0.1537539	-0.5424689	-0.8803293
2	0.0109057	0.2799487	-0.6604012	-1.3020012
3	0.0112094	0.2180631	-0.6085926	-1.1062971
4	0.0112844	0.1062400	-0.4811037	-0.6939712
5	0.0132973	0.1383752	-0.5257446	-0.8268663
6	0.0151261	0.2826436	-0.6648921	-1.3172451
7	0.0158631	0.2586521	-0.6463401	-1.2446946
8	0.0163434	0.2328090	-0.6248264	-1.1634345
9	0.0164393	0.2606048	-0.6482612	-1.2517610

Orbit	m	p ₁	X(L ₃)	C
10	0.0174247	0.1986083	−0.5940092	−1.0518603
11	0.0180814	0.0716630	−0.4279560	−0.5521557
12	0.0184429	0.0675920	−0.4202738	−0.5331581
13	0.0191575	0.0671578	−0.4199114	−0.5324152
14	0.0199390	0.3114354	−0.6890029	−1.4110138
15	0.0251782	0.0924640	−0.4692184	−0.6610185
16	0.0255297	0.0597152	−0.4090195	−0.5076828
17	0.0259616	0.0950651	−0.4738906	−0.6738777
18	0.0268613	0.0567309	−0.4034510	−0.4949934
19	0.0277300	0.0850640	−0.4587414	−0.6329721
20	0.0296614	0.0394231	−0.3623404	−0.4057785
21	0.0333746	0.0548514	−0.4039212	−0.4980308
22	0.0319611	0.0583232	−0.4106074	−0.5131799
23	0.0386483	0.0346688	−0.3551057	−0.3954530
24	0.0425825	0.0347341	−0.3582760	−0.4037709
25	0.0429769	0.0882056	−0.4744354	−0.6764349

Table 4: Non-symmetric doubly asymptotic 1–8 orbits ($\delta = -0.001$)

Orbit	m	p ₁	X(L ₃)	C
1	0.0101619	0.0025921	−0.1463043	−0.0769863
2	0.0102566	0.0444562	−0.3617517	−0.3964028
3	0.0104734	0.0484394	−0.3721250	−0.4189337
4	0.0131897	0.0125615	−0.2431417	−0.1883774
5	0.0138884	0.0187698	−0.2766951	−0.2394133
6	0.0148987	0.0680671	−0.4187290	−0.5286447
7	0.0174691	0.0174549	−0.2732385	−0.2365868
8	0.0175029	0.0714384	−0.4271223	−0.5499658
9	0.0178751	0.0012398	−0.1235849	−0.0706428
10	0.0179866	0.0188525	−0.2803150	−0.2482211
11	0.0185890	0.1627650	−0.5576207	−0.9277875
12	0.0189814	0.1593535	−0.5540629	−0.9160333
13	0.0190662	0.0096190	−0.2283872	−0.1738474
14	0.0190859	0.0012178	−0.1240529	−0.0726960
15	0.0215885	0.0911523	−0.4646618	−0.6484027
16	0.0249372	0.0614268	−0.4122425	−0.5151853
17	0.0274538	0.0923393	−0.4705623	−0.6648147
18	0.0283621	0.0810802	−0.4523102	−0.6160724
19	0.0322693	0.1337526	−0.5322106	−0.8440823
20	0.0336104	0.0490149	−0.3903854	−0.4674765

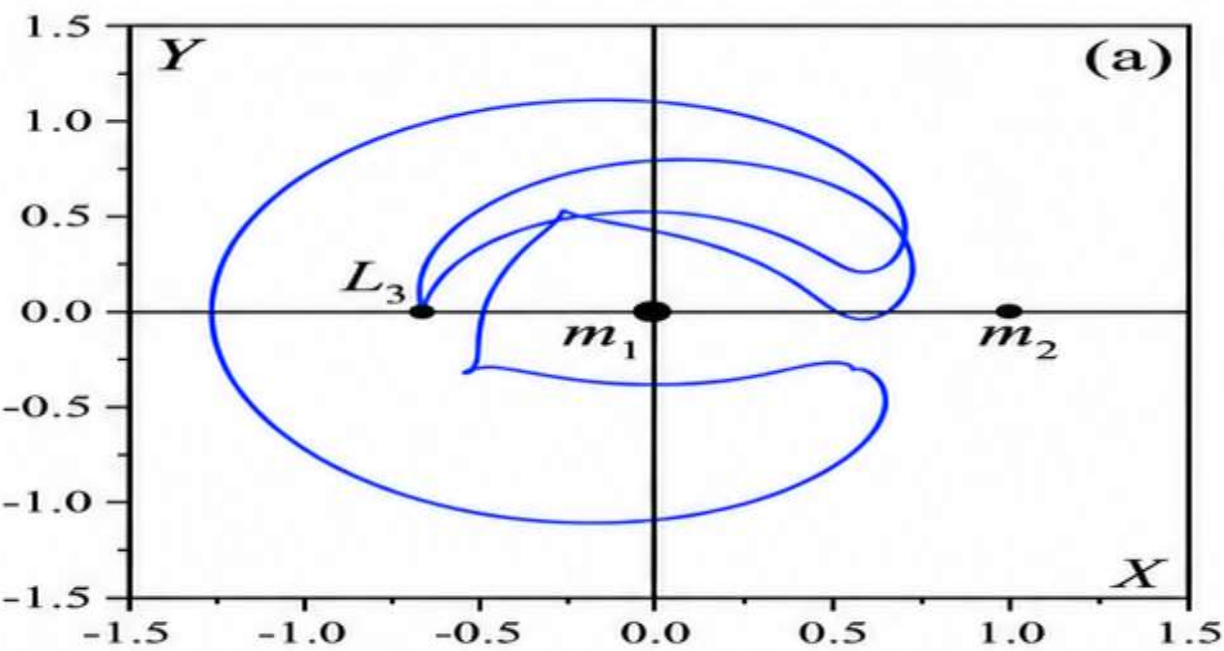
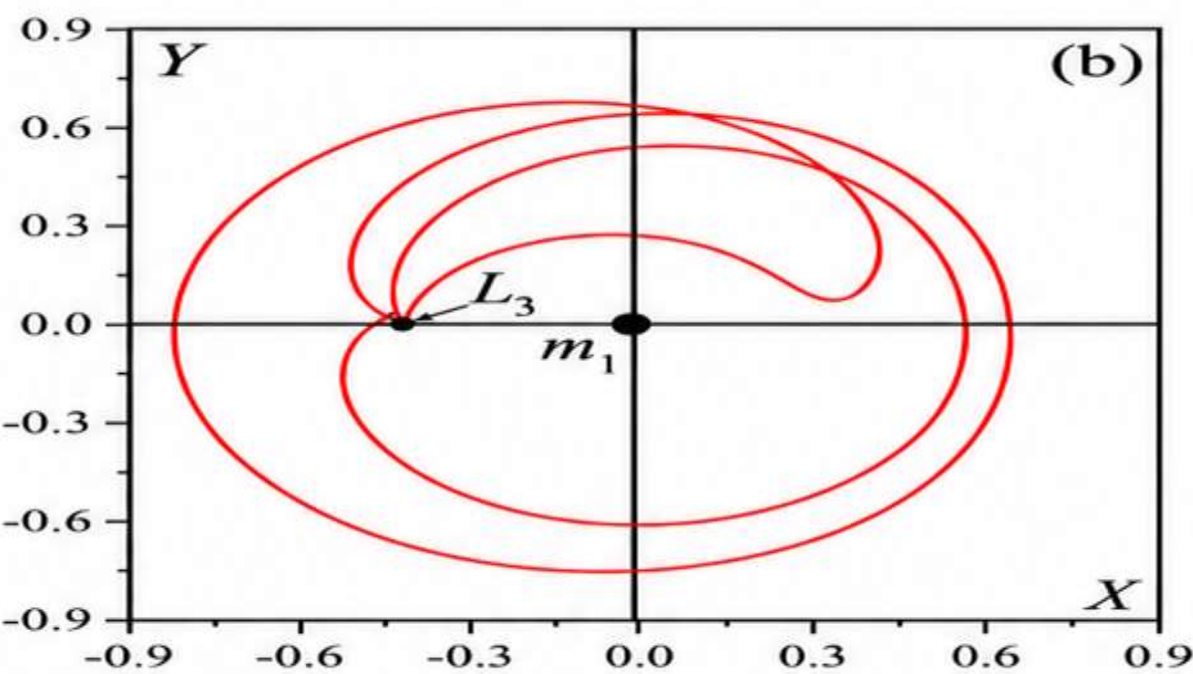
Orbit	m	p_1	$X(L_3)$	C
21	0.0350130	0.1186093	-0.5142846	-0.7894537
22	0.0364953	0.1161430	-0.5118988	-0.7822882
23	0.0419255	0.1108221	-0.5080920	-0.7708755
24	0.0421045	0.0311411	-0.3465591	-0.3806142
25	0.0423451	0.0303592	-0.3441579	-0.3761603
26	0.0425498	0.0291425	-0.3402033	-0.3688008
27	0.0427694	0.0348186	-0.3586746	-0.4046604
28	0.0441982	0.0082854	-0.2392031	-0.2124102
29	0.0444893	0.0723482	-0.4477320	-0.6067409

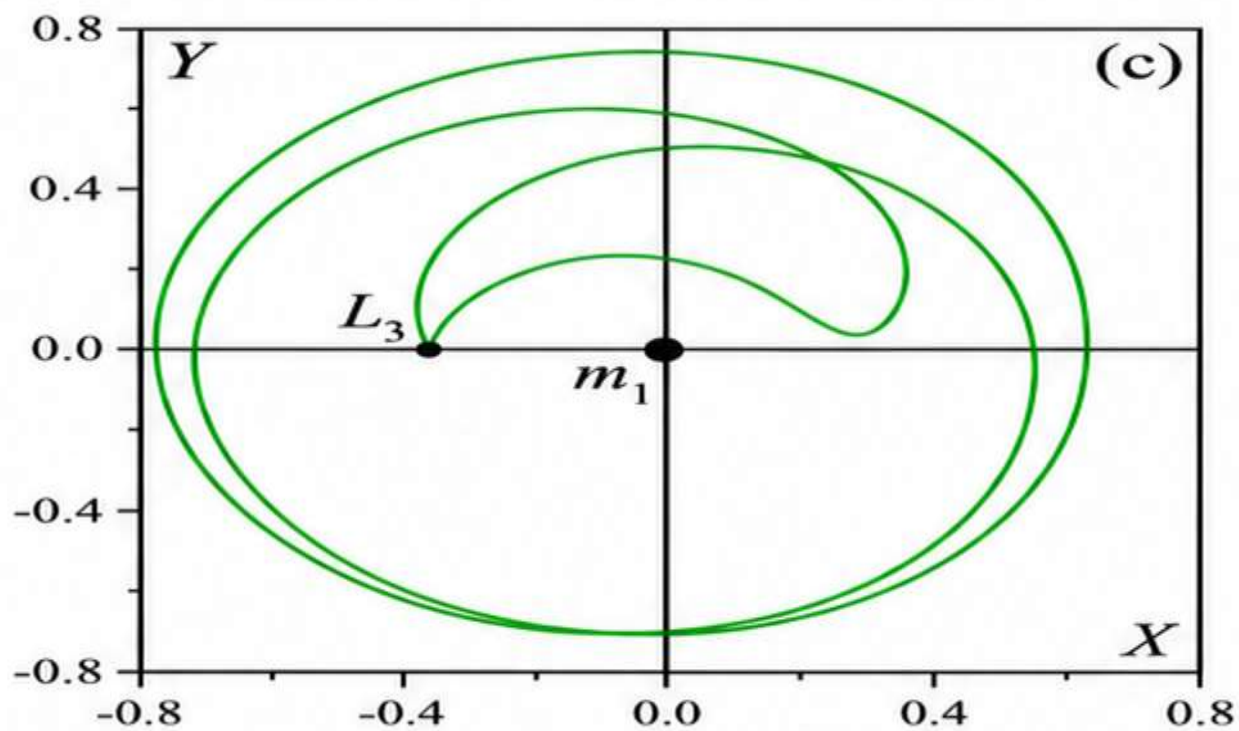
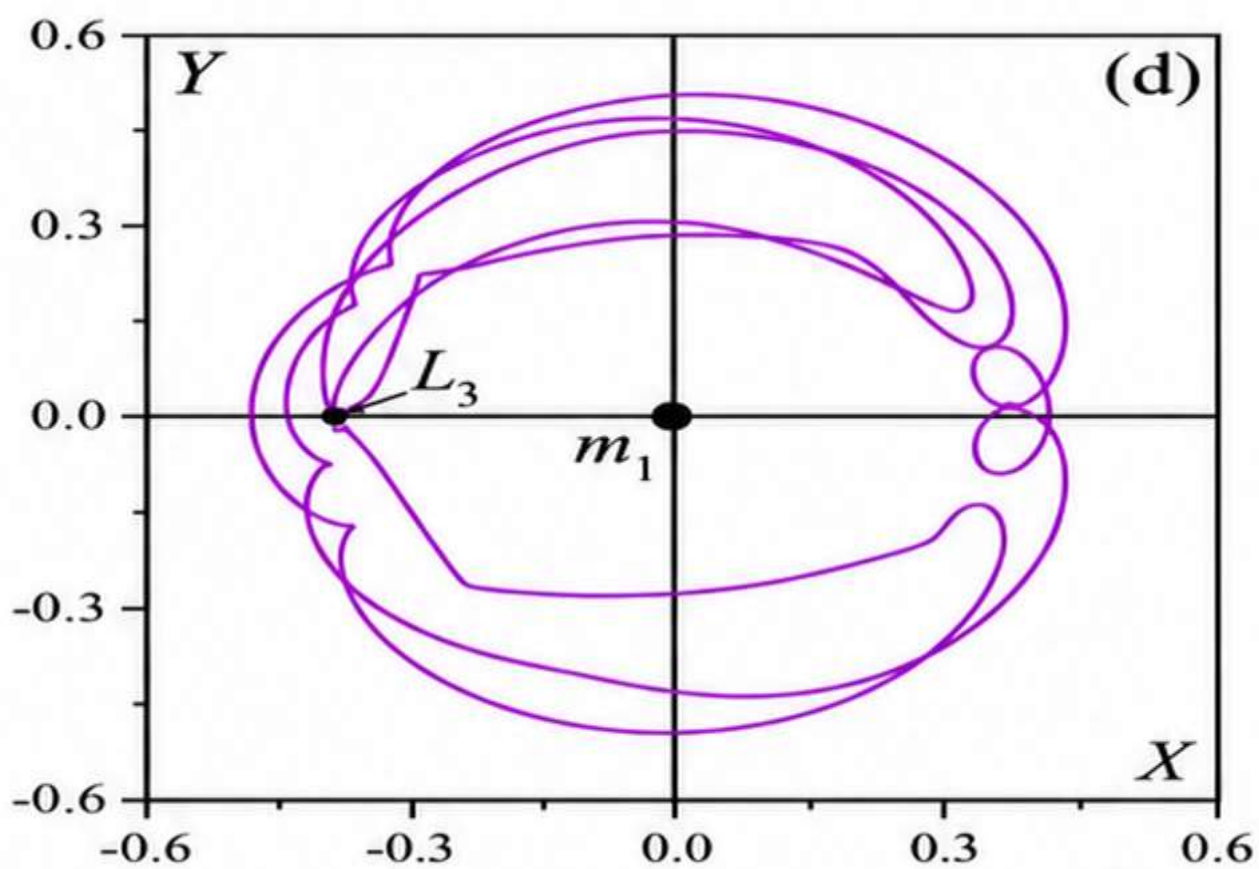
Table 5: Non-symmetric doubly asymptotic 1–10 orbits ($\delta = -0.001$)

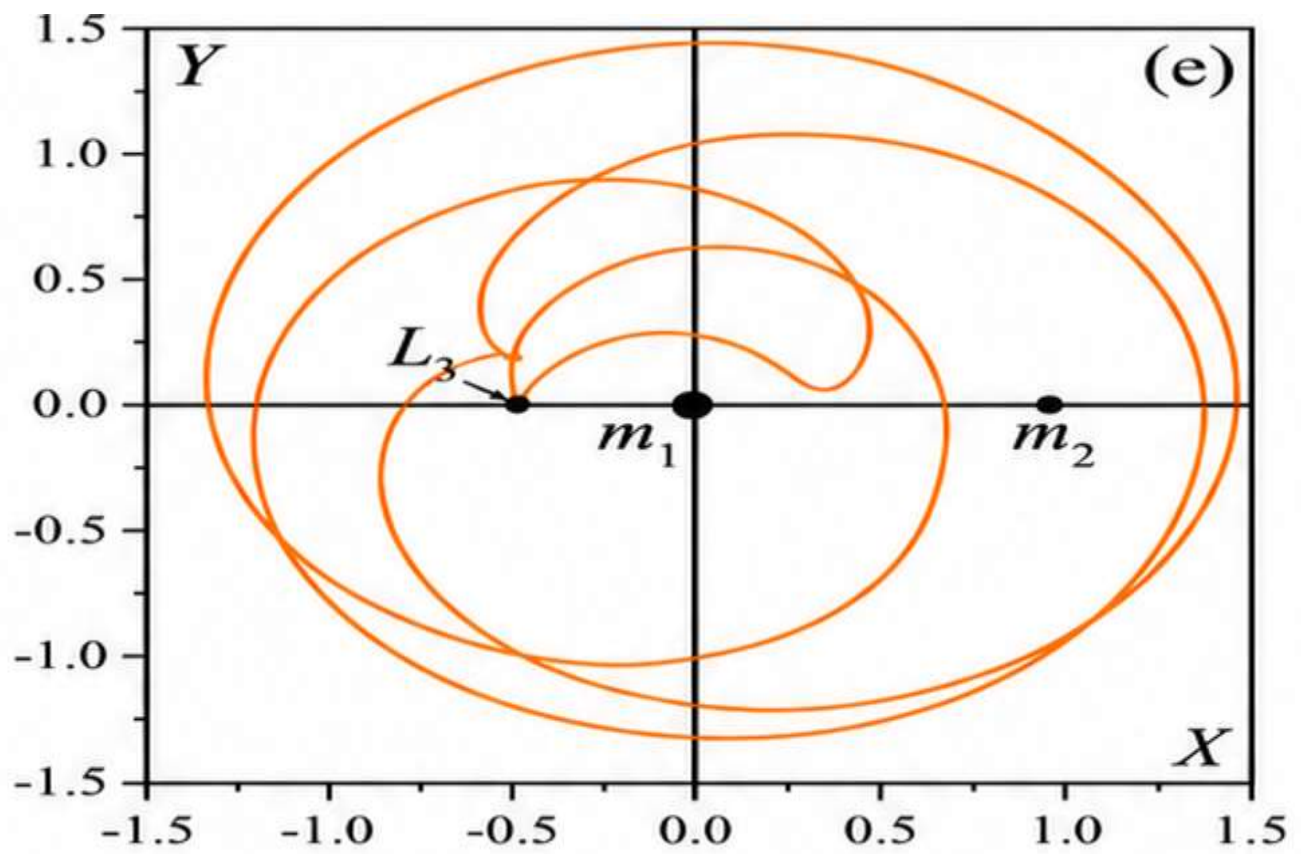
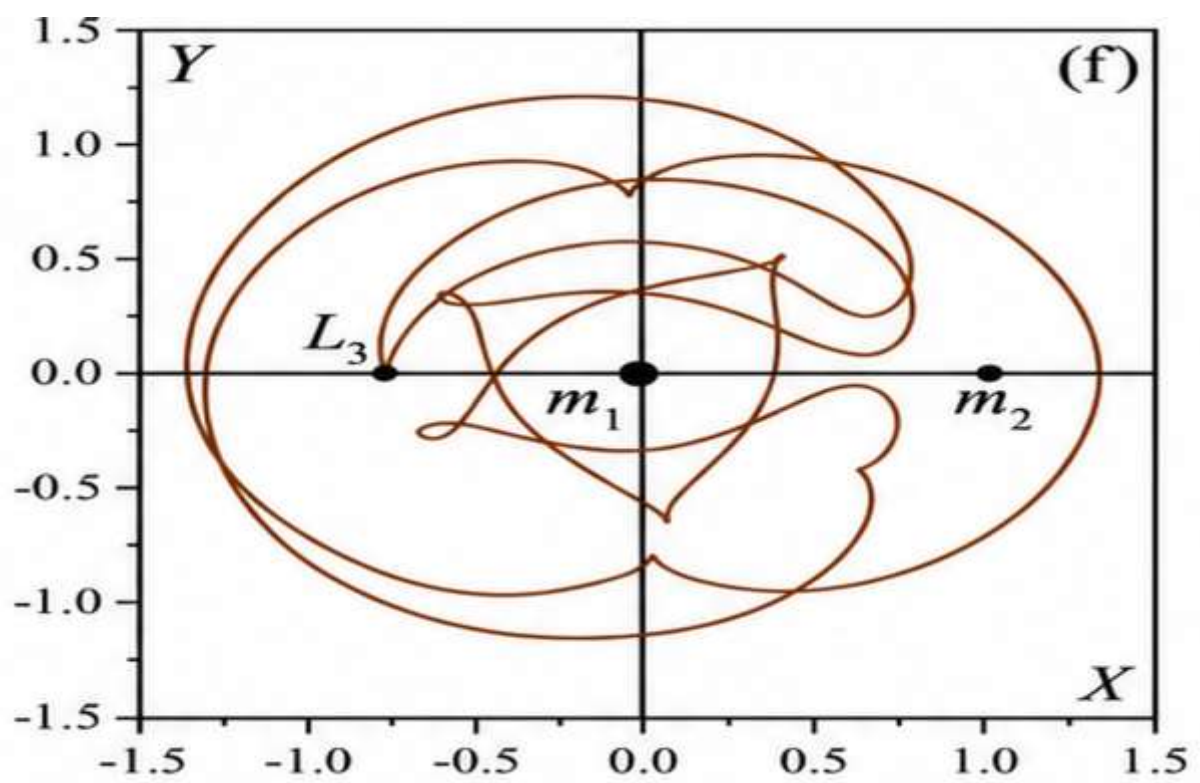
Orbit	m	p_1	$X(L_3)$	C
1	0.0103778	0.0591312	-0.3969605	-0.4752893
2	0.0105103	0.0499067	-0.3757730	-0.4269945
3	0.0106116	0.0639771	-0.4074272	-0.5002215
4	0.0110459	0.0199100	-0.2796909	-0.2422284
5	0.0116006	0.3291742	-0.6968894	-1.4489648
6	0.0116962	0.4032983	-0.7450384	-1.6556044
7	0.0120629	0.0268739	-0.3087649	-0.2929065
8	0.0121940	0.3822034	-0.7322642	-1.5990171
9	0.0130626	0.0168622	-0.2667981	-0.2231942

Orbit	m	p ₁	X(L ₃)	C
10	0.0144530	0.0185126	−0.2759328	−0.2386053
11	0.0144855	0.3610379	−0.7199379	−1.5440351
12	0.0169570	0.3716659	−0.7281484	−1.5774880
13	0.0170188	0.0067776	−0.2035548	−0.1415518
14	0.0175731	0.0101231	−0.2307610	−0.1755209
15	0.0180553	0.0114781	−0.2402919	−0.1887161
16	0.0190067	0.0244621	−0.3049626	−0.2904010
17	0.0190369	0.0458981	−0.3719348	−0.4216470
18	0.0194019	0.0427472	−0.3639119	−0.4046402
19	0.0195903	0.0271025	−0.3153772	−0.3093664
20	0.0211005	0.0062509	−0.2020376	−0.1441914
21	0.0245134	0.0434781	−0.3696328	−0.4188596
22	0.0266083	0.0087104	−0.1804800	−0.2278332

Figure 4. Non-symmetric doubly asymptotic orbits to L_3 : (a)–(c) show 1–6 orbits (entries 7, 21 and 24 of Table 3); (d) and (e) show 1–8 orbits (entries 3 and 29 of Table 4); (f) shows a 1–10 orbit (entry 8 of Table 5).



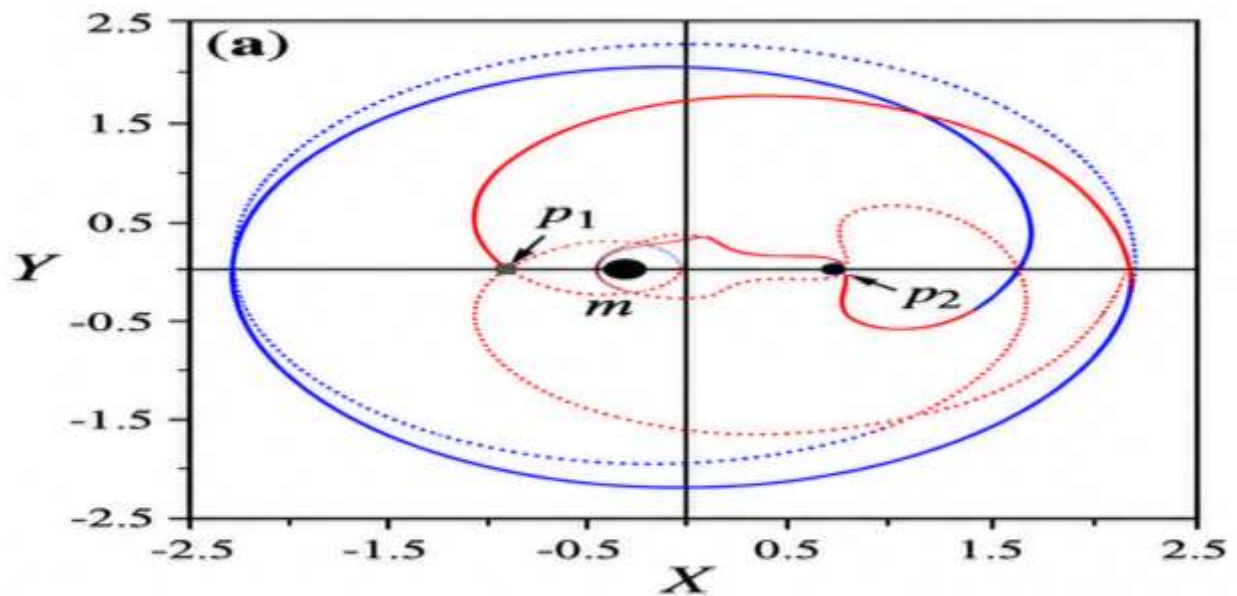




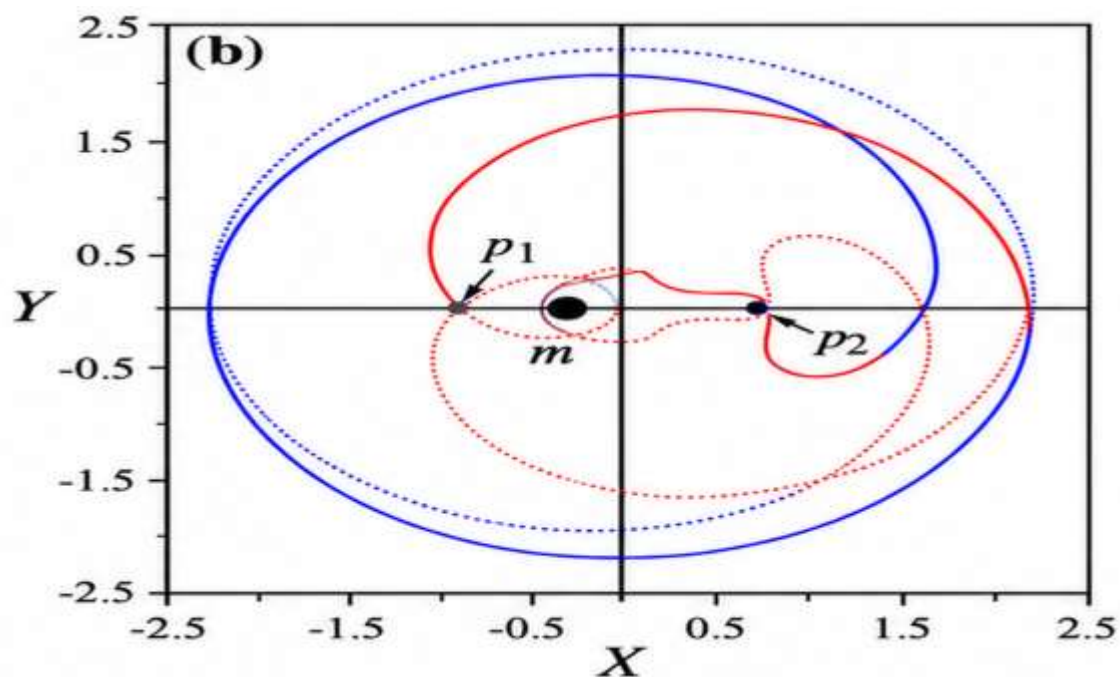
3.2 Symmetric periodic solutions and orbit families

The paired I–IV and II–III asymptotic branches are associated with neighbouring symmetric periodic solutions. The article tabulates initial position X_0 , initial transverse velocity $\dot{Y}_0$, the half-period intersection coordinate $X(T/2)$, Jacobi constant C , and the half-period $T/2$. Limiting simple periodic orbits are recorded separately. Empty limiting-orbit cells are retained where the published table provides no entry.

Figure 5. (a) An I–IV homoclinic asymptotic orbit together with its symmetric II–III orbit (entry 12 of Table 2).



(b) The symmetric periodic solution generated by these asymptotic orbits.



(c) Their behaviour near L_3 in the upper half of the XY-plane; asymptotic orbits are solid and the periodic solution is dotted.

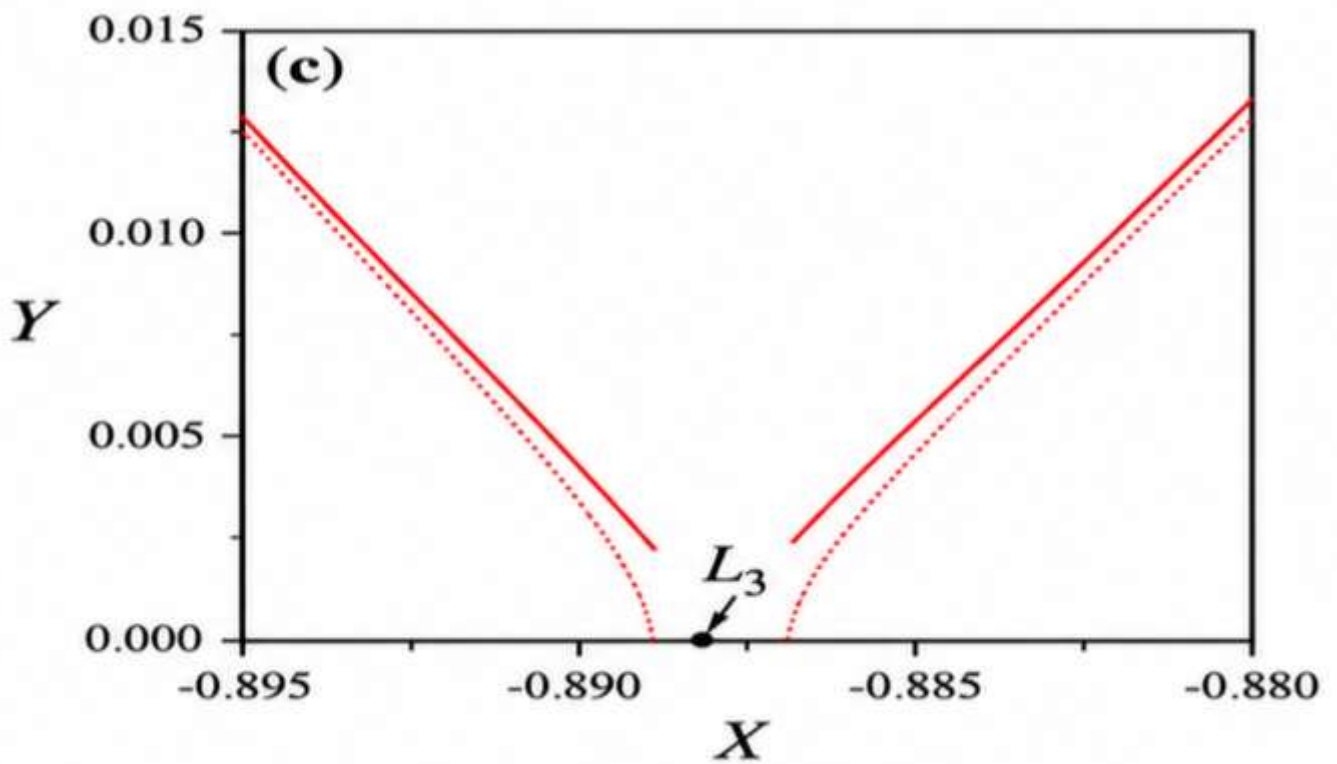


Table 6. Families containing periodic orbits associated with the asymptotic solutions in Figure 3, and limiting simple periodic orbits.

S et	N o.	X_0	$\dot{Y}_0$	$X(T/2)$	C	T/2	Limit X_0	Limit $\dot{Y}_0$	Limit $X(T/2)$	Limit T/2
a	1	-0.0193 239	-0.0002 722	-0.0193 092	-0.0233 711	132.5288 456	-0.0193 181	-0.0002 808	-0.0195 888	3.0454 767
a	2	-0.0194 636	0.00001 49	-0.0194 422	-0.0233 711	129.7238 043	-	-	-	-
a	3	-0.0195 913	0.00028 22	-0.0193 090	-0.0233 711	135.5743 443	-0.0195 890	0.00027 88	-0.0193 179	3.0455 777
b	1	-0.0336 570	0.00167 22	-0.0315 278	-0.0405 974	78.40287 77	-0.0336 495	0.00166 15	-0.0320 518	2.9888 404
b	2	-0.0328 618	0.00001 48	-0.0318 793	-0.0405 983	77.82264 13	-	-	-	-
b	3	-0.0336 551	0.00166 95	-0.0315 278	-0.0405 974	79.22485 20	-0.0336 495	0.00166 15	-0.0320 518	2.9888 394
c	1	-0.1771 154	0.08656 14	-0.0872 278	-0.0711 042	72.08246 34	-0.1767 190	0.08609 02	-0.0874 547	3.0559 828
c	2	-0.1324 314	0.00014 82	-0.1319 517	-0.0735 726	73.85078 08	-	-	-	-
c	3	-0.1717 671	0.07680 71	-0.0917 458	-0.0716 485	78.25944 04	-0.1716 029	0.07660 81	-0.0926 680	3.0515 201
d	1	-0.5778 179	0.32491 52	-0.2355 995	-0.4720 938	55.48161 91	-0.5746 069	0.32119 95	-0.2409 901	3.0437 657
d	2	-0.4099 662	0.00148 31	-0.3993 444	-0.5078 111	49.27010 55	-	-	-	-
d	3	-0.5778 927	0.32655 98	-0.2332 610	-0.4710 788	55.01310 01	-0.5768 630	0.32537 26	-0.2386 754	3.0443 455
e	1	-0.9561 714	0.17508 39	-0.9520 907	-2.0681 311	31.10276 10	-0.9553 243	0.17399 01	-0.8181 466	2.5560 100
e	2	-0.8889 569	0.00141 89	-0.8869 671	-2.0827 706	31.93086 59	-	-	-	-
e	3	-0.9582 616	0.18042 93	-0.8156 237	-2.0671 913	33.65873 82	-0.9574 116	0.17933 49	-0.8159 024	2.5561 692

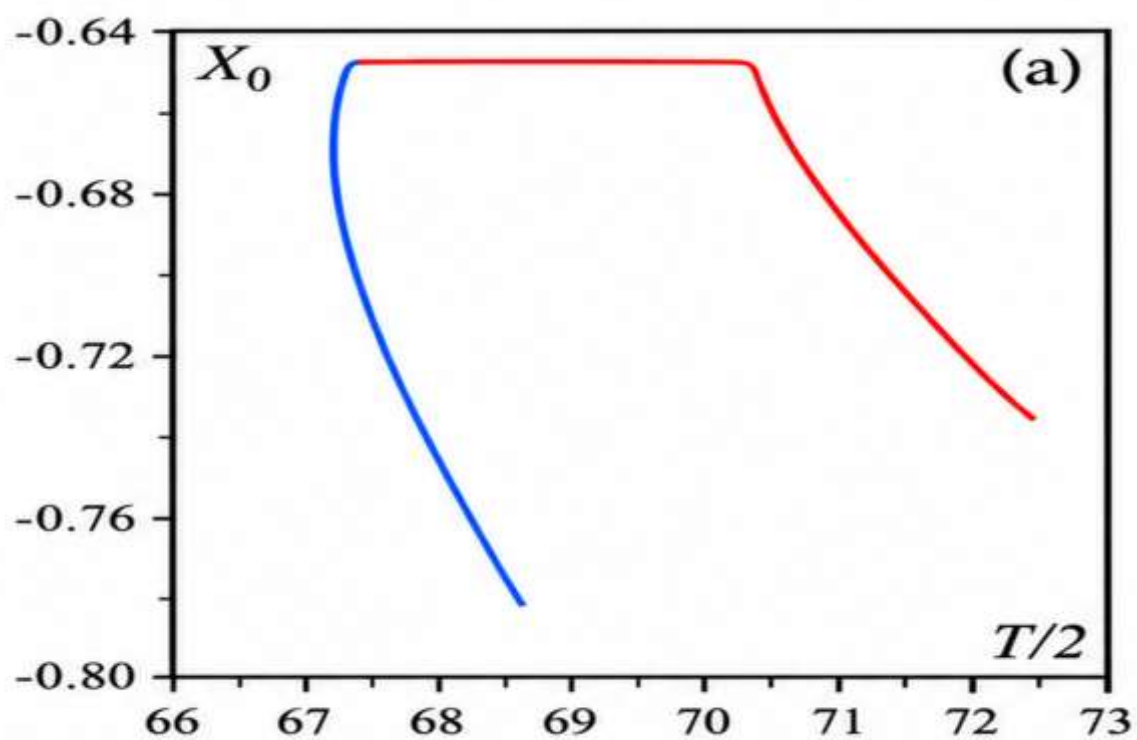
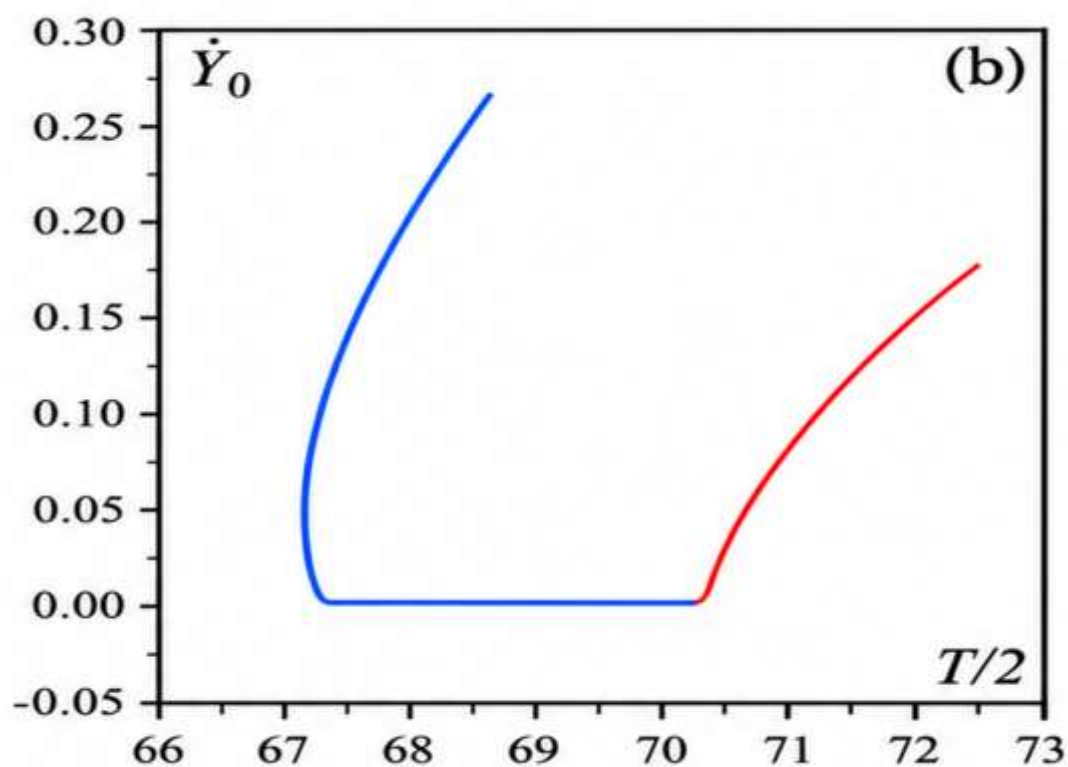
Note. The column “Set” identifies the groups of periodic-orbit solutions, labelled a, b and c, while “No.” identifies the individual periodic-orbit entries within each group. The limiting-

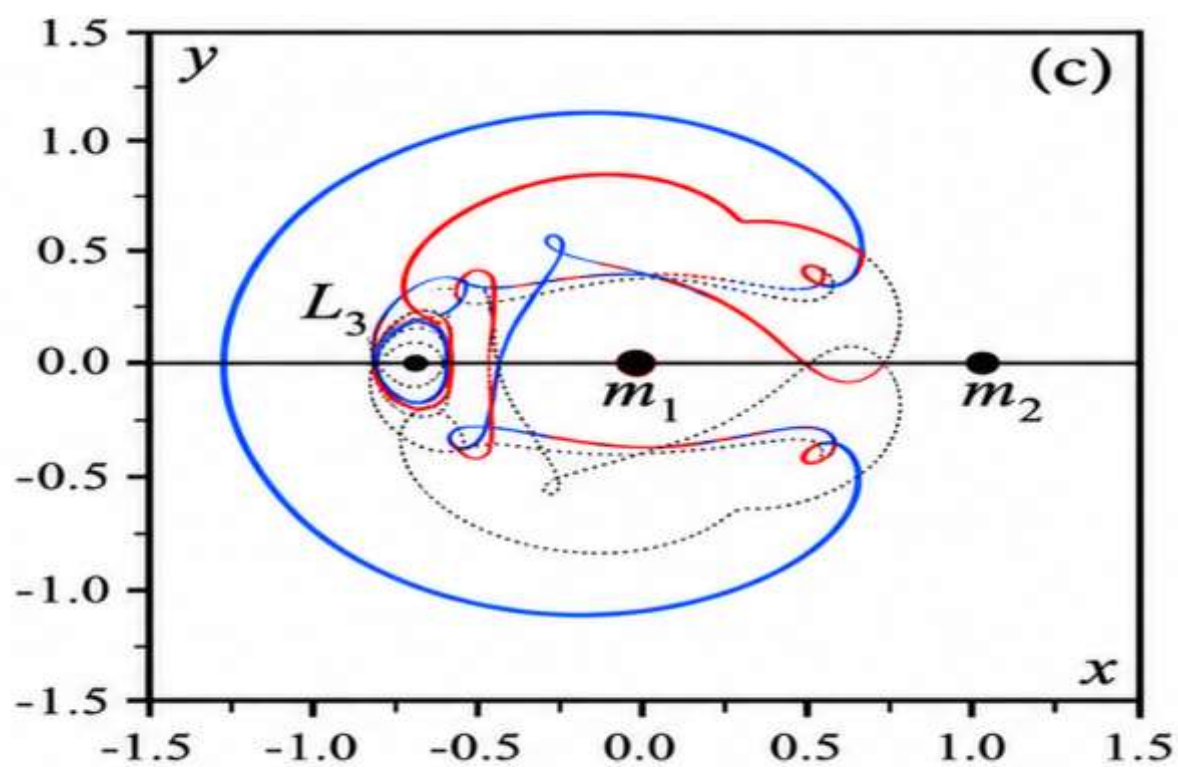
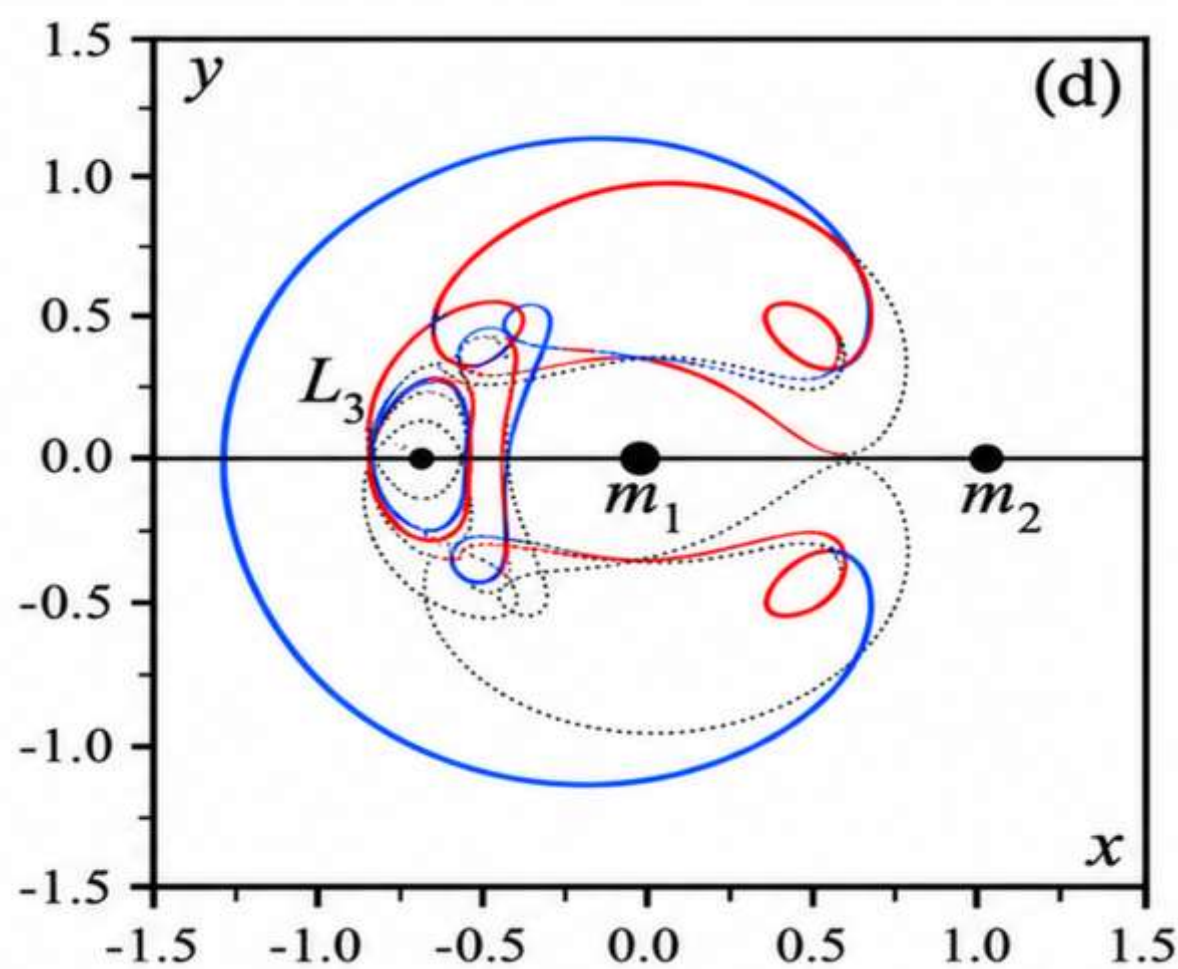
orbit columns report the corresponding initial conditions, half-period intersection coordinates and half-periods, where available.

Table 7. Families containing periodic orbits associated with the asymptotic solutions in Figure 4, and limiting simple periodic orbits.

S et	N o.	X_0	$\dot{Y}_0$	$X(T/2)$	C	$T/2$	Limit X_0	Limit $\dot{Y}_0$	Limit $X(T/2)$	Limit $T/2$
a	1	-0.7825 042	0.26509 13	-0.4157 337	-1.2240 611	68.58399 72	-0.7815 216	0.26381 58	-0.5106 908	3.0863 625
a	2	-0.6473 089	0.00150 15	-0.4893 835	-1.2446 952	67.52499 33	-	-	-	-
a	3	-0.7359 532	0.17682 01	-0.4525 598	-1.2357 985	72.45324 18	-0.7353 900	0.17605 72	-0.5570 858	3.0853 165
b	1	-0.5768 954	0.33510 79	-0.2113 621	-0.4588 568	49.62116 22	-0.5723 649	0.32993 62	-0.2321 848	3.0216 019
b	2	-0.4049 019	0.00147 17	-0.3499 867	-0.4980 316	51.85575 88	-	-	-	-
b	3	-0.5161 419	0.22295 71	-0.2662 584	-0.4813 627	55.63590 67	-0.5148 976	0.22143 25	-0.2914 279	3.0056 983
c	1	-0.4038 598	0.09413 78	-0.3123 465	-0.4009 003	48.00983 60	-0.4026 859	0.09257 03	-0.3134 617	2.9526 412
c	2	-0.3592 591	0.00146 42	-0.3549 601	-0.4037 718	47.97542 98	-	-	-	-
c	3	-0.4238 618	0.13492 78	-0.2911 924	-0.3975 491	52.96262 98	-0.4234 779	0.13443 65	-0.2922 087	2.9563 806
d	1	-0.4690 724	0.18598 35	-0.2624 486	-0.4088 371	124.1156 416	-0.4669 560	0.18330 00	-0.2769 387	3.0931 243
d	2	-0.3730 650	0.00151 78	-0.3426 775	-0.4189 341	127.0342 798	-	-	-	-
d	3	-0.4577 841	0.16574 20	-0.2727 996	-0.4108 811	124.3320 083	-0.4569 454	0.16466 39	-0.2870 185	3.0924 369
e	1	-0.2942 347	-0.3698 177	-0.6239 395	-0.5746 056	54.78024 62	-0.2961 485	-0.3654 808	-0.5961 526	2.9801 297
e	2	-0.4467 178	-0.0014 841	-0.4855 285	-0.6067 420	52.66641 47	-	-	-	-
e	3	-0.5641 221	0.23643 24	-0.5907 921	-0.5869 566	55.86511 57	-0.5647 169	0.23715 94	-0.3287 403	2.9723 977
f	1	-0.8807 289	0.28956 01	-0.9699 810	-1.5742 688	101.5826 678	-0.8826 784	0.29209 97	-0.5814 941	3.1019 336
f	2	-0.7312 256	-0.0015 102	-0.8515 586	-1.5990 181	98.59233 32	-	-	-	-
f	3	-0.8893 602	0.30508 26	-0.9778 120	-1.5717 019	102.1275 976	-0.8901 987	0.30616 94	-0.5739 377	3.1020 569

Figure 6. Family of periodic orbits containing the solution obtained from the asymptotic orbit in Figure 4(a) (entry 7 of Table 3): (a) and (b) characteristic curves ($T/2, X_0$) and ($T/2, \dot{Y}_0$); (c) and (d) termination orbits.





4. Discussion

The numerical results illustrate the relationship between local asymptotic dynamics and global orbital structures near L_3 in the planar photogravitational restricted three-body problem. The governing equations and fourth-order asymptotic expansions provide initial approximations for the outgoing and incoming motions represented by the four branches in Figures 1 and 2. Numerical integration and phase-space matching describe non-symmetric homoclinic trajectories that depart from and return asymptotically to L_3 . MATLAB-based graphical analysis displays the asymptotic trajectories and zero-velocity curves, supporting their geometric interpretation. This behaviour is consistent with Kalantonis et al. (2006), while the distinction between local dynamics and global invariant structures accords with the analysis of collinear equilibrium regions by Jorba and Masdemont (1999).

The five orbit classes in Tables 1–5 exhibit different homoclinic return geometries, from 1–2 to 1–10 crossing configurations. Figures 3 and 4 show increasingly elaborate paths involving multiple excursions before approach to the equilibrium. Variations in the mass parameter m , radiation reduction factor p_1 , equilibrium coordinate $X(L_3)$ and Jacobi constant C correspond to differences in the reported orbital configurations. For example, the 1–4 class spans appreciable differences in

$X(L_3)$ and C . However, because m and p_1 vary simultaneously, their separate quantitative effects cannot be inferred from these comparisons.

Figure 5 illustrates the association between the symmetry-related I–IV and II–III homoclinic branches and a neighbouring symmetric periodic orbit. Tables 6 and 7 record initial conditions, half-periods and limiting solutions of the corresponding periodic families. The reported continuation approaches limiting simple periodic orbits of the L_3 Lyapunov family, linking the homoclinic configurations with the organization of neighbouring periodic motion. This interpretation is consistent with the broader role of periodic solutions in restricted three-body dynamics discussed by Gómez and Mondelo (2001).

The characteristic curves in Figure 6 illustrate variations in orbital initial conditions and half-periods along a continued periodic family. Together with the graphical representations of the trajectory classes, they provide a means of analysing relationships among orbital solutions rather than isolated paths alone. Connecting trajectories and invariant manifolds also organize nonlinear motion in the classical restricted three-body problem (Koon et al., 2000). Those findings nevertheless concern a different dynamical setting and cannot be transferred directly to the present photogravitational configurations.

These interpretations are limited to the planar, circular and conservative photogravitational model. The available numerical configurations illustrate distinct orbital geometries and associated periodic families, but do not establish their stability or persistence under additional perturbations. Independent convergence and manifold-transversality analyses are also not provided. Accordingly, the equation formulation and graphical analysis characterize the relationships among the reported asymptotic branches, homoclinic configurations and periodic families near L_3 without establishing broader dynamical properties.

5. Conclusion

This study examines non-symmetric homoclinic trajectories and associated symmetric periodic-orbit families near L_3 in the planar photogravitational restricted three-body problem. A formulation based on rotating-frame equations, with radiation-reduction factors and fourth-order asymptotic expansions, describes outgoing and incoming trajectories. Numerical and graphical analysis, including MATLAB representations of the orbital configurations, examines five crossing-based classes (1–2, 1–4, 1–6, 1–8 and 1–10) and their relationships with neighbouring symmetric periodic solutions approaching the L_3 Lyapunov family. The analysed configurations illustrate differences in orbital geometry, equilibrium location and Jacobi

constant, but do not isolate the effects of individual parameters or establish stability and manifold transversality. Further independent numerical verification, convergence studies and perturbation analyses are needed to assess the robustness of these connections.

6. Recommendations

- I. Quantify numerical convergence and matching residuals across the reported orbit classes to strengthen reproducibility.
- II. Evaluate monodromy-based stability and manifold geometry along the continued periodic families.
- III. Examine the persistence of L_3 connections under two luminous primaries and additional physically relevant perturbations.

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